

# A CHAIN LEVEL BATALIN-VILKOVISKY STRUCTURE IN STRING TOPOLOGY VIA DE RHAM CHAINS

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**ABSTRACT.** The aim of this paper is to define a chain level refinement of the Batalin-Vilkovisky (BV) algebra structure on homology of the free loop space of a closed  $C^\infty$ -manifold. Namely, we propose a new chain model of the free loop space, and define an action of a certain chain model of the framed little disks operad on it, recovering the original BV structure on homology level. We also compare this structure to a solution of Deligne's conjecture for Hochschild cochain complexes of differential graded algebras. To define the chain model of the loop space, we introduce a notion of de Rham chains, which is a hybrid of singular chains and differential forms.

## 1. INTRODUCTION

Let us begin with the following facts:

- (a): For any differential graded algebra  $A$ , the Hochschild cohomology  $H^*(A, A)$  has a Gerstenhaber algebra structure.
- (b): Let  $M$  be a closed, oriented  $d$ -dimensional  $C^\infty$ -manifold,  $\mathcal{L}M := C^\infty(S^1, M)$  be the free loop space. Then,  $\mathbb{H}_*(\mathcal{L}M) := H_{*+d}(\mathcal{L}M)$  has a Batalin-Vilkovisky (in particular, Gerstenhaber) algebra structure.
- (c): Let  $\mathcal{A}_M$  denote the differential graded algebra of differential forms on  $M$ . There exists a linear map  $\mathbb{H}_*(\mathcal{L}M : \mathbb{R}) \rightarrow H^*(\mathcal{A}_M, \mathcal{A}_M)$  defined by iterated integrals of differential forms, which preserves the Gerstenhaber structures.

(a) is originally due to Gerstenhaber [15]. (b) is due to Chas-Sullivan [4], which is the very first paper on string topology. (c) relates the geometric construction (b) to the algebraic construction (a). It seems that (c) is also well-known to specialists (see Remark 1.4).

(a)–(c) concern algebraic structures on homology level, and it is an important and interesting problem to define chain level refinements of these structures. For (a), so called Deligne's conjecture claims that a certain chain model of the little disks operad acts on the Hochschild cochain complex. Various affirmative solutions to this conjecture and its variations are known; see [26] Part I Section 1.19, [24] Section 13.3.15, and the references therein.

The aim of this paper is to propose a chain level algebraic structure which lifts (b) (the Batalin-Vilkovisky (BV) algebra structure in string topology), and compare it with a solution to Deligne's conjecture via a chain map which is a chain level lift of (c).

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Let us briefly describe our main result (see Theorem 1.5 for the rigorous statement). First of all, for any closed, oriented  $C^\infty$ -manifold  $M$ , we define a chain complex  $C_*^{\mathcal{L}^M}$  over  $\mathbb{R}$ , which is a chain model of  $\mathcal{L}M$ . We also define a differential graded operad  $f\tilde{\Lambda}$  and its suboperad  $\tilde{\Lambda}$ . These operads are chain models of the framed little disks operad and the little disks operad, and defined independently from  $M$ . We establish the following statements:

- (a'): For any differential graded algebra  $A$ , the Hochschild complex  $C^*(A, A)$  admits an action of  $\tilde{\Lambda}$ , which lifts the Gerstenhaber structure on  $H^*(A, A)$  in (a).
- (b'): For any closed, oriented  $C^\infty$ -manifold  $M$ , the chain complex  $C_*^{\mathcal{L}^M}$  admits an action of  $f\tilde{\Lambda}$ , thus  $\mathbb{H}_*(C_*^{\mathcal{L}^M})$  has the BV algebra structure. There exists an isomorphism  $\Phi : \mathbb{H}_*(\mathcal{L}M) \cong H_*(C_*^{\mathcal{L}^M})$  preserving the BV structures.
- (c'): There exists a  $\tilde{\Lambda}$ -equivariant chain map  $J : C_*^{\mathcal{L}^M} \rightarrow C^*(\mathcal{A}_M, \mathcal{A}_M)$ , such that  $H_*(J) \circ \Phi : \mathbb{H}_*(\mathcal{L}M) \rightarrow H^*(\mathcal{A}_M, \mathcal{A}_M)$  coincides the map in (c).

There may be several different ways to work out chain level structures in string topology, based on choices of chain models of the free loop space. The singular chain complex has the transversality trouble, namely string topology operations are defined only for chains transversal to each other. The Hochschild complex of differential forms (used e.g. in [25]) avoids this trouble, however it is not always a correct chain model of the free loop space (see Remark 1.8), and loses some geometric informations (e.g. lengths of loops, see Section 1.5.3). Our chain model  $C_*^{\mathcal{L}^M}$  is an intermediate one of these two.

This section is organized as follows. In Sections 1.1–1.4, we recall several basic definitions and facts, fixing various notations and signs. In Section 1.5, we state Theorem 1.5, which is our main result, and a few supplementary results. Section 1.6 discusses previous works and 1.7 discusses potential applications to symplectic topology. Section 1.8 explains the plan of the rest of this paper.

## 1.1. Preliminaries from operads.

1.1.1. *Operads.* First we briefly recall the notion of (nonsymmetric) operads. The main aim is to fix conventions, and we refer [26] Part II Section 1.2 for details.

Let  $\mathcal{C}$  be any symmetric monoidal category with a multiplication  $\times$  and a unit  $1_{\mathcal{C}}$ . A *nonsymmetric operad*  $\mathcal{P}$  in  $\mathcal{C}$  consists of the following data:

- An object  $\mathcal{P}(n)$  for every integer  $n \geq 0$ .
- A morphism  $\circ_i : \mathcal{P}(n) \times \mathcal{P}(m) \rightarrow \mathcal{P}(n + m - 1)$  for every  $1 \leq i \leq n$  and  $m \geq 0$ . These morphisms are called (*partial*) *compositions*.
- A morphism  $1_{\mathcal{P}} : 1_{\mathcal{C}} \rightarrow \mathcal{P}(1)$  called a *unit* of  $\mathcal{P}$ .

We require that compositions satisfy associativity, and  $1_{\mathcal{P}}$  is a two-sided unit for compositions. When  $\mathcal{P}(n)$  admits a right action of the symmetric group  $\mathbb{S}_n$  ( $\mathbb{S}_0$  is the trivial group) for each  $n \geq 0$ , such that these actions are compatible with compositions,  $\mathcal{P}$  is called an *operad* in  $\mathcal{C}$ .

For any (nonsymmetric) operads  $\mathcal{P}$  and  $\mathcal{Q}$ , a morphism of (nonsymmetric) operads  $\varphi : \mathcal{P} \rightarrow \mathcal{Q}$  is a sequence of morphisms  $(\varphi(n) : \mathcal{P}(n) \rightarrow \mathcal{Q}(n))_{n \geq 0}$  which preserves the above structures. When  $\varphi(n)$  are monics for all  $n \geq 0$ , we say that  $\mathcal{P}$  is a suboperad of  $\mathcal{Q}$ .

1.1.2. *Graded and dg operads.* Throughout this paper, all vector spaces are defined over  $\mathbb{R}$ . A graded vector space  $V$  is a sequence  $(V_n)_{n \in \mathbb{Z}}$  of vector spaces. A differential graded (or dg) vector space (or chain complex) is a pair  $(V, \partial)$  of a graded vector space  $V$  and  $\partial : V_* \rightarrow V_{*-1}$  satisfying  $\partial^2 = 0$ . We may consider any graded vector space as a dg vector space with  $\partial = 0$ . One can define the symmetric monoidal structure on the category of dg vector spaces as follows:

$$(V \otimes W)_n := \bigoplus_{i+j=n} V_i \otimes W_j,$$

$$\partial(v \otimes w) := \partial v \otimes w + (-1)^{|v|} v \otimes \partial w,$$

$$V \otimes W \rightarrow W \otimes V; v \otimes w \mapsto (-1)^{|v||w|} w \otimes v.$$

The unit  $\bar{\mathbb{R}}$  is defined as  $\bar{\mathbb{R}}_n := \begin{cases} \mathbb{R} & (n = 0) \\ 0 & (n \neq 0) \end{cases}$  and  $\partial = 0$ . In this paper, we mainly work on the category of graded and dg vector spaces. Operads in these categories are called *graded operads* and *dg operads*, respectively.

For any dg vector spaces  $V$  and  $W$ ,  $\text{Hom}(V, W)$  has the structure of a dg vector space:

$$\text{Hom}(V, W)_n := \prod_{k \in \mathbb{Z}} \text{Hom}(V_k, W_{k+n}), \quad (\partial f)(v) := \partial(f(v)) - (-1)^{|f|} f(\partial v).$$

For any dg vector space  $V$ ,  $\text{End}(V) := (\text{Hom}(V^{\otimes n}, V))_{n \geq 0}$  has the structure of a dg operad defined as follows ( $f \in \text{Hom}(V^{\otimes n}, V)$ ,  $g \in \text{Hom}(V^{\otimes m}, V)$ , and  $\sigma \in \mathbb{S}_n$ ):

$$\begin{aligned} (f \circ_i g)(v_1 \otimes \cdots \otimes v_{n+m-1}) &:= (-1)^{|g|(|v_1| + \cdots + |v_{i-1}|)} f(v_1 \otimes \cdots \otimes g(v_i \otimes \cdots \otimes v_{i+m-1}) \otimes \cdots \otimes v_{n+m-1}), \\ 1_{\text{End}(V)} &:= \text{id}_V, \\ (f^\sigma)(v_1 \otimes \cdots \otimes v_n) &:= \prod_{\substack{i < j \\ \sigma(i) > \sigma(j)}} (-1)^{|v_i||v_j|} f(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}). \end{aligned}$$

This dg operad is called the *endomorphism operad* of  $V$ .

For any dg operad  $\mathcal{P}$ , a dg  $\mathcal{P}$ -algebra is a chain complex  $V$  with a morphism  $\mathcal{P} \rightarrow \text{End}(V)$  of dg operads. For each  $n \geq 0$  we have a chain map

$$\mathcal{P}(n) \otimes V^{\otimes n} \rightarrow V; \quad x \otimes v_1 \otimes \cdots \otimes v_n \mapsto x \cdot (v_1 \otimes \cdots \otimes v_n).$$

For any dg  $\mathcal{P}$ -algebras  $V$  and  $W$ , a chain map  $\varphi : V \rightarrow W$  is called a morphism of dg  $\mathcal{P}$ -algebras if  $\varphi(x \cdot (v_1 \otimes \cdots \otimes v_n)) = x \cdot (\varphi(v_1) \otimes \cdots \otimes \varphi(v_n))$  for any  $x \in \mathcal{P}(n)$  and  $v_1, \dots, v_n \in V$ . For any graded operad  $\mathcal{P}$ , the notions of graded  $\mathcal{P}$ -algebras and their morphisms are defined in a similar way.

For any dg operad  $\mathcal{P} = (\mathcal{P}(n))_{n \geq 0}$ , a *dg ideal* of  $\mathcal{P}$  is a sequence  $\mathcal{Q} = (\mathcal{Q}(n))_{n \geq 0}$  such that the following conditions hold:

- For every  $n \geq 0$ ,  $\mathcal{Q}(n)$  is a chain subcomplex of  $\mathcal{P}(n)$ , which is preserved by the  $\mathbb{S}_n$ -action on  $\mathcal{P}(n)$ .
- For any  $x \in \mathcal{P}(n)$ ,  $y \in \mathcal{P}(m)$  and  $1 \leq i \leq n$ ,

$$x \in \mathcal{Q}(n) \quad \text{or} \quad y \in \mathcal{Q}(m) \implies x \circ_i y \in \mathcal{Q}(n+m-1).$$

For any dg ideal  $\mathcal{Q} \subset \mathcal{P}$ , the quotient  $\mathcal{P}/\mathcal{Q} := (\mathcal{P}(n)/\mathcal{Q}(n))_{n \geq 0}$  has a natural structure of a dg operad, and there exists a natural morphism of dg operads  $\mathcal{P} \rightarrow \mathcal{P}/\mathcal{Q}$ . For any graded operad  $\mathcal{P}$ , the notions of its graded ideals and associated quotient graded operads are defined in the obvious way (see [24] Section 5.2.14).

**1.1.3. Gerstenhaber and Batalin-Vilkovisky (BV) operads.** Gerstenhaber and BV operads are graded operads, which play central roles in this paper. We recall definitions of these operads using generators and relations, partially following [24] Sections 13.3.12 and 13.7.4.

The *Gerstenhaber operad*  $\mathcal{G}$  is generated by  $a \in \mathcal{G}(2)_0$ ,  $b \in \mathcal{G}(2)_1$ ,  $u \in \mathcal{G}(0)_0$  with the following relations:

- (a):  $a^{(12)} = a$ ,  $a \circ_1 a = a \circ_2 a$ .
- (b):  $b^{(12)} = b$ ,  $b \circ_1 b + (b \circ_1 b)^{(123)} + (b \circ_1 b)^{(321)} = 0$ .
- (ab):  $b \circ_1 a = a \circ_2 b + (a \circ_1 b)^{(23)}$ .
- (u):  $a \circ_1 u = a \circ_2 u = 1_{\mathcal{G}}$ ,  $b \circ_1 u = b \circ_2 u = 0$ .

More precisely, we consider the free operad  $E$  (see [24] Section 5.5) generated by  $a$ ,  $b$ ,  $u$ , and a graded ideal  $R \subset E$  generated by relations (a), (b), (ab), (u). Then, we define  $\mathcal{G} := E/R$ .

For any graded  $\mathcal{G}$ -algebra  $V$ , we define operations  $\bullet$  and  $\{, \}$  on  $V$  as

$$v \bullet w := a \cdot (v \otimes w), \quad \{v, w\} := (-1)^{|v|} b \cdot (v \otimes w).$$

Then,  $(V, \bullet)$  is a graded commutative, associative algebra, and  $(V, \{, \})$  is a graded Lie algebra (with grading shifted by 1). The triple  $(V, \bullet, \{, \})$  is called a *Gerstenhaber algebra*.

**Remark 1.1.** In the above definition, any Gerstenhaber algebra  $V$  has a unit  $u \cdot 1 \in V_0$  of the multiplication  $\bullet$ . It seems that the existence of a unit is usually not assumed.

The *BV operad*  $\mathcal{BV}$  is generated by  $a \in \mathcal{BV}(2)_0$ ,  $b \in \mathcal{BV}(2)_1$ ,  $\Delta \in \mathcal{BV}(1)_1$ ,  $u \in \mathcal{BV}(0)_0$  with the relations (a), (b), (ab), (u) and

$$\Delta \circ_1 \Delta = 0, \quad b = \Delta \circ_1 a - a \circ_1 \Delta - a \circ_2 \Delta, \quad \Delta \circ_1 u = 0.$$

Obviously, there exists a natural morphism of graded operads  $\mathcal{G} \rightarrow \mathcal{BV}$ .

For any graded  $\mathcal{BV}$ -algebra  $V$ , we define an operation  $\Delta$  on  $V$  by  $\Delta(v) := \Delta \cdot v$ . The triple  $(V, \bullet, \Delta)$  is called a *BV algebra*. The bracket  $\{, \}$  is recovered by the formula  $\{v, w\} = (-1)^{|v|} \Delta(v \bullet w) - (-1)^{|v|} \Delta v \bullet w - v \bullet \Delta w$ .

For any integer  $r \geq 1$ , let  $f\mathcal{D}(r)$  be the set of tuples  $(D_1, \dots, D_r, z_1, \dots, z_r)$  such that

- For each  $1 \leq i \leq r$ ,  $D_i$  is a closed disk of positive radius contained in  $\{z \in \mathbb{C} \mid |z| \leq 1\}$ . We denote its center by  $p_i$ .  $z_i$  is a point on  $\partial D_i$ .
- $D_1, \dots, D_r$  are disjoint.

The set  $f\mathcal{D}(r)$  has a natural topology. Let  $\mathcal{D}(r)$  denote the subspace of  $f\mathcal{D}(r)$ , which consists of  $(D_1, \dots, D_r, z_1, \dots, z_r)$  such that  $z_i - p_i \in \mathbb{R}_{>0}$  for every  $1 \leq i \leq r$ . We define  $f\mathcal{D}(0) = \mathcal{D}(0)$  to be the space consists of a point. Then,  $f\mathcal{D} = (f\mathcal{D}(r))_{r \geq 0}$  has a natural structure of a topological operad, and  $\mathcal{D} = (\mathcal{D}(r))_{r \geq 0}$  is its suboperad.  $\mathcal{D}$  (resp.  $f\mathcal{D}$ ) is called the *little disks* (resp. *framed little disks*) operad. There are isomorphisms of graded operads  $H_*(\mathcal{D}) \cong \mathcal{G}$  ([10]) and  $H_*(f\mathcal{D}) \cong \mathcal{BV}$  ([17]), which are compatible with the inclusion maps.

**1.2. Gerstenhaber structure on Hochschild cohomology.** A *differential graded associative algebra* is a dg vector space  $A$  with a degree 0 product  $A \otimes A \rightarrow A$ , which is associative, and satisfies the Leibniz rule. We also assume that it has a unit  $1_A \in A_0$ .

**Remark 1.2.** We abbreviate the term “differential graded associative algebra” as “dga algebra”. Here the letter “a” stands for “associative”, not for algebra (see [24] pp. 29).

Let  $A$  be any dga algebra. A dg  $A$ -bimodule is a dg vector space  $M$  with degree 0 left and right  $A$ -actions  $A \otimes M \rightarrow M$  and  $M \otimes A \rightarrow M$ , which satisfy the Leibniz rule and associativity.

For every  $k \geq 1$  and  $i = 0, \dots, k$ , we define a chain map  $\delta_{k,i} : \text{Hom}_*(A^{\otimes k-1}, M) \rightarrow \text{Hom}_*(A^{\otimes k}, M)$  by

$$\delta_{k,i}(f)(a_1 \otimes \dots \otimes a_k) := \begin{cases} (-1)^{|a_1||f|} a_1 \cdot f(a_2 \otimes \dots \otimes a_k) & (i = 0), \\ f(a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_k) & (1 \leq i \leq k-1), \\ f(a_1 \otimes \dots \otimes a_{k-1}) \cdot a_k & (i = k). \end{cases}$$

We set  $\delta_k : \text{Hom}_*(A^{\otimes k-1}, M) \rightarrow \text{Hom}_*(A^{\otimes k}, M)$  by  $\delta_k(f) := (-1)^{|f|+k-1} \sum_{i=0}^k (-1)^i \delta_{k,i}(f)$ ,

and define the Hochschild cochain complex  $C^*(A, M) := \left( \prod_{k=0}^{\infty} \text{Hom}_{*+k}(A^{\otimes k}, M), \partial_{\text{Hoch}} \right)$

by  $\partial_{\text{Hoch}}(f_k)_{k \geq 0} := (\partial f_k)_{k \geq 0} + (\delta_k(f_{k-1}))_{k \geq 1}$ . Notice that  $\partial_{\text{Hoch}}$  decreases the degree by 1. The cohomology of this complex is denoted by  $H^*(A, M)$ , and called the *Hochschild cohomology*.

Notice that  $A$  itself has a natural structure of a dg  $A$ -bimodule.  $C^*(A, A)$  has natural dga and dg Lie algebra structures, with operations  $\circ$  and  $\{, \}$  defined below. The product  $\circ$  is defined as

$$(f \circ g)_k(a_1 \otimes \dots \otimes a_k) := \sum_{l+m=k} (-1)^{(|a_1|+\dots+|a_l|+l)(|g|+m)} f_l(a_1 \otimes \dots \otimes a_l) g_m(a_{l+1} \otimes \dots \otimes a_k).$$

The bracket  $\{, \}$  is defined as

$$\{f, g\} := f * g - (-1)^{(|f|+1)(|g|+1)} g * f,$$

where  $*$  is defined as

$$(f * g)_k(a_1 \otimes \cdots \otimes a_k) := \sum_{\substack{l+m=k+1 \\ 1 \leq i \leq l}} (-1)^\dagger f_l(a_1 \otimes \cdots \otimes a_{i-1} \otimes g_m(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_k),$$

$$\dagger := (|g| + m)(|a_1| + \cdots + |a_{i-1}| + i - 1) + (|g| + 1)(l - i).$$

The operations  $\circ$  and  $\{, \}$  induce the Gerstenhaber structure on  $H^*(A, A)$ . This result is originally due to Gerstenhaber [15].

**1.3. BV structure in string topology.** Throughout this paper, we set  $S^1 := \mathbb{R}/\mathbb{Z}$ . Let  $M$  be a closed, oriented  $C^\infty$ -manifold of dimension  $d$ .  $\mathcal{L}M := C^\infty(S^1, M)$  is equipped with the  $C^\infty$ -topology. We often abbreviate  $\mathcal{L}M$  as  $\mathcal{L}$ . Also, we often use the notation  $\mathbb{H}_*(\cdot) := H_{*+d}(\cdot)$ . In [4], Chas-Sullivan introduced the *loop product* on  $\mathbb{H}_*(\mathcal{L}) = H_{*+d}(\mathcal{L})$ . Let us briefly recall its definition.

Let us consider the evaluation map  $e : \mathcal{L} \rightarrow M; \gamma \mapsto \gamma(0)$ , and the fiber product

$$\mathcal{L} \times_e \mathcal{L} := \{(\gamma, \gamma') \in \mathcal{L}^{\times 2} \mid \gamma(0) = \gamma'(0)\}.$$

Let  $U$  be a tubular neighborhood of  $\mathcal{L} \times_e \mathcal{L} \subset \mathcal{L}^{\times 2}$ , and  $H_*(U, U \setminus \mathcal{L} \times_e \mathcal{L}) \cong H_{*-d}(\mathcal{L} \times_e \mathcal{L})$  be the Thom isomorphism. The Gysin map  $H_*(\mathcal{L}^{\times 2}) \rightarrow H_{*-d}(\mathcal{L} \times_e \mathcal{L})$  is defined as the composition of the following maps:

$$H_*(\mathcal{L}^{\times 2}) \rightarrow H_*(\mathcal{L}^{\times 2}, \mathcal{L}^{\times 2} \setminus \mathcal{L} \times_e \mathcal{L}) \cong H_*(U, U \setminus \mathcal{L} \times_e \mathcal{L}) \cong H_{*-d}(\mathcal{L} \times_e \mathcal{L}).$$

Let  $c : \mathcal{L} \times_e \mathcal{L} \rightarrow \mathcal{L}$  denote the concatenation map. Precisely, it is defined as follows (see the remark on pp. 780 [12]). Let us take an increasing  $C^\infty$ -function  $\nu : [0, 1] \rightarrow [0, 1]$  such that  $\nu(t) = t$  and  $\nu^{(m)}(t) = 0$  ( $\forall m \geq 1$ ) ( $\nu^{(m)}$  denotes the  $m$ -th derivative) for any  $t \in \{0, 1/2, 1\}$ . Then,  $c : \mathcal{L} \times_e \mathcal{L} \rightarrow \mathcal{L}$  is defined by

$$c(\gamma_1, \gamma_2)(t) := \begin{cases} \gamma_1(2\nu(t)) & (0 \leq t \leq 1/2), \\ \gamma_2(2\nu(t) - 1) & (1/2 \leq t \leq 1). \end{cases}$$

It is easy to see that  $H_*(c) : H_*(\mathcal{L} \times_e \mathcal{L}) \rightarrow H_*(\mathcal{L})$  does not depend on choices of  $\nu$ .

The loop product  $\bullet : \mathbb{H}_*(\mathcal{L})^{\otimes 2} \rightarrow \mathbb{H}_*(\mathcal{L})$  is defined as the composition of the following three maps. The first map is the cross product and the second map is the Gysin map.

$$\mathbb{H}_*(\mathcal{L})^{\otimes 2} \xrightarrow{\times} H_{*+2d}(\mathcal{L}^{\times 2}) \longrightarrow \mathbb{H}_*(\mathcal{L} \times_e \mathcal{L}) \xrightarrow{\mathbb{H}_*(c)} \mathbb{H}_*(\mathcal{L}).$$

Let us consider the map  $i_M : M \rightarrow \mathcal{L}M; p \mapsto \text{constant loop at } p$ . Let  $\cap : \mathbb{H}_*(M)^{\otimes 2} \rightarrow \mathbb{H}_*(M)$  denote the intersection product. Then,

$$\mathbb{H}(i_M)(x) \bullet \mathbb{H}_*(i_M)(y) = \mathbb{H}_*(i_M)(x \cap y) \quad (\forall x, y \in \mathbb{H}_*(M)).$$

On the other hand,  $\mathcal{L}$  admits a natural  $S^1$ -action  $r : S^1 \times \mathcal{L} \rightarrow \mathcal{L}$ , which is defined by  $r(t, \gamma)(\theta) := \gamma(\theta - t)$ . We define  $\Delta : \mathbb{H}_*(\mathcal{L}) \rightarrow \mathbb{H}_{*+1}(\mathcal{L})$  by  $\Delta(x) := -H_*(r)([S^1] \times x)$ , where  $[S^1] \in H_1(S^1)$  is represented by the singular chain  $\Delta^1 \rightarrow S^1; t \mapsto [t]$  (see the next subsection for the definition of  $\Delta^1$ ).

**Theorem 1.3** ([4], [12], [5]). *For any closed, oriented  $d$ -dimensional  $C^\infty$ -manifold  $M$ ,  $(\mathbb{H}_*(\mathcal{L}M), \bullet, \Delta)$  is a BV algebra.*

This result is the starting point of string topology. The bracket  $\{ , \}$  of this BV structure is called the *loop bracket*.

**1.4. Iterated integrals of differential forms.** There is a relation between Gerstenhaber structures on loop space homology (Theorem 1.3) and Hochschild cohomology (Section 1.2). We explain this relation via iterated integrals of differential forms, which originates in [6].

To discuss iterated integrals of differential forms, it is convenient to work with  $C^\infty$ -singular chains on  $\mathcal{L}M$ . Let us define the  $k$ -dimensional simplex  $\Delta^k$  by

$$\Delta^k := \begin{cases} \mathbb{R}^0 & (k = 0), \\ \{(t_1, \dots, t_k) \in \mathbb{R}^k \mid 0 \leq t_1 \leq \dots \leq t_k \leq 1\}. & (k \geq 1). \end{cases}$$

A map  $\sigma : \Delta^k \rightarrow \mathcal{L}M$  is said to be of class  $C^\infty$ , if there exists an open neighborhood  $U$  of  $\Delta^k \subset \mathbb{R}^k$  and a map  $\bar{\sigma} : U \rightarrow \mathcal{L}M$ , such that  $\bar{\sigma}|_U = \sigma$ , and  $U \times S^1 \rightarrow M; (u, \theta) \mapsto \bar{\sigma}(u)(\theta)$  is of class  $C^\infty$ . Let  $C_k^{\text{sm}}(\mathcal{L}M)$  denote the  $\mathbb{R}$ -vector space generated by all  $C^\infty$ -maps  $\Delta^k \rightarrow \mathcal{L}M$ . It is easy to see that, any  $C^\infty$ -map  $\sigma : \Delta^k \rightarrow \mathcal{L}M$  is continuous with respect to the  $C^\infty$ -topology on  $\mathcal{L}M$ . Therefore,  $C_*^{\text{sm}}(\mathcal{L}M)$  is a subcomplex of the singular chain complex of  $\mathcal{L}M$ . In Section 4, we show that this inclusion map is a quasi-isomorphism (Theorem 4.1). Therefore,  $H_*^{\text{sm}}(\mathcal{L}M) := H_*(C_*^{\text{sm}}(\mathcal{L}M)) \cong H_*(\mathcal{L}M)$ .

For any  $j \in \mathbb{Z}$ , let us define

$$\mathcal{A}^j(M) := \begin{cases} \mathbb{R}\text{-vector space of } C^\infty \text{ } j\text{-forms on } M & (0 \leq j \leq d = \dim M), \\ 0 & (\text{otherwise}). \end{cases}$$

Then,  $(\mathcal{A}^{-*}(M), d, \wedge)$  is a dga algebra, where  $d$  denotes the exterior derivative, and  $\wedge$  denotes the exterior product. We denote it by  $\mathcal{A}_M$ . We define a dg  $\mathcal{A}_M$ -bimodule structure on  $\mathcal{A}_M^\vee[d]_* := \text{Hom}(\mathcal{A}^{*+d}(M), \mathbb{R})$  as follows:

$$\begin{aligned} (\partial\varphi)(\alpha) &:= (-1)^{|\varphi|+1}\varphi(d\alpha), \\ (\alpha \cdot \varphi)(\beta) &:= (-1)^{|\alpha||\varphi|}\varphi(\alpha \wedge \beta), \quad (\varphi \cdot \alpha)(\beta) := \varphi(\alpha \wedge \beta). \end{aligned}$$

A morphism of  $\mathcal{A}_M$ -bimodules  $\mathcal{A}_M \rightarrow \mathcal{A}_M^\vee[d]; \alpha \mapsto (\beta \mapsto \int_M \alpha \wedge \beta)$  is a quasi-isomorphism (this is an obvious consequence of the Poincaré duality).

For any  $C^\infty$ -map  $\sigma : \Delta^l \rightarrow \mathcal{L}M$  and  $i = 0, \dots, k$ , we define  $\sigma_{k,i} : \Delta^l \times \Delta^k \rightarrow M$  by

$$\sigma_{k,i}(x, t_1, \dots, t_k) := \begin{cases} \sigma(x)(0) & (i = 0) \\ \sigma(x)(t_i) & (1 \leq i \leq k), \end{cases}$$

and define  $I_k(\sigma) \in \text{Hom}(\mathcal{A}_M^{\otimes k}, \mathcal{A}_M^\vee[d])$  by

$$I_k(\sigma)(\eta_1 \otimes \dots \otimes \eta_k)(\eta_0) := (-1)^{(2d+2k+l+1)l/2} \int_{\Delta^l \times \Delta^k} \sigma_{k,1}^* \eta_1 \wedge \dots \wedge \sigma_{k,k}^* \eta_k \wedge \sigma_{k,0}^* \eta_0.$$

It is easy to see that

$$(1) \quad I : C_{*+d}^{\text{sm}}(\mathcal{L}M) \rightarrow C^*(\mathcal{A}_M, \mathcal{A}_M^\vee[d]); \quad \sigma \mapsto (I_k(\sigma))_k$$

is a chain map (signs are checked in Section 6.3). Taking homology, we obtain a map

$$(2) \quad \mathbb{H}_*(\mathcal{L}M) \rightarrow H^*(\mathcal{A}_M, \mathcal{A}_M^\vee[d]) \cong H^*(\mathcal{A}_M, \mathcal{A}_M).$$

This map preserves the Gerstenhaber structures on  $\mathbb{H}_*(\mathcal{L}M)$  and  $H^*(\mathcal{A}_M, \mathcal{A}_M)$ .

**Remark 1.4.** The fact that (2) preserves the Gerstenhaber structures seems to be known; see [25] for the product, and [14] Section 7 for the bracket. We can recover this fact as a consequence of Theorem 1.5, see Remark 1.7.

**1.5. Summary of results.** We state our main result Theorem 1.5 and some supplementary results.

**1.5.1. Main result.** Let us recall some notations:  $\mathcal{G}$  and  $\mathcal{BV}$  denote the Gerstenhaber and BV operads. For any  $C^\infty$ -manifold  $M$  of dimension  $d$ ,  $\mathcal{L}M := C^\infty(S^1, M)$ , and  $\mathbb{H}_*(\mathcal{L}M) := H_{*+d}(\mathcal{L}M : \mathbb{R})$ .  $\mathcal{A}_M$  denotes the dga algebra of differential forms on  $M$ .

**Theorem 1.5.** *There exists a dg operad  $f\tilde{\Lambda}$  and its suboperad  $\tilde{\Lambda}$ , satisfying the following properties.*

- (i): *There exist isomorphisms of graded operads  $H_*(f\tilde{\Lambda}) \cong \mathcal{BV}$  and  $H_*(\tilde{\Lambda}) \cong \mathcal{G}$ , such that the following diagram commutes:*

$$\begin{array}{ccc} H_*(\tilde{\Lambda}) & \longrightarrow & H_*(f\tilde{\Lambda}) \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{G} & \longrightarrow & \mathcal{BV}. \end{array}$$

- (ii): *For any dga algebra  $A$ , the Hochschild complex  $C^*(A, A)$  has a dg  $\tilde{\Lambda}$ -algebra structure, which lifts the Gerstenhaber structure on  $H^*(A, A)$ .*
- (iii): *For any closed, oriented  $C^\infty$ -manifold  $M$ , there exist:*
- (a): *A dg  $f\tilde{\Lambda}$ -algebra  $C_*^{\mathcal{L}M}$ .*
  - (b): *An isomorphism  $\Phi : \mathbb{H}_*(\mathcal{L}M) \cong H_*(C_*^{\mathcal{L}M})$  of BV algebras.*
  - (c): *A morphism of dg  $\tilde{\Lambda}$ -algebras  $J : C_*^{\mathcal{L}M} \rightarrow C^*(\mathcal{A}_M, \mathcal{A}_M)$  such that  $H_*(J) \circ \Phi : \mathbb{H}_*(\mathcal{L}M) \rightarrow H^*(\mathcal{A}_M, \mathcal{A}_M)$  is equal to the map (2).*

**Remark 1.6.** It seems that (although there are some technical issues the author has not confirmed) the following stronger version of (i) is also true: there exists a zig-zag of quasi-isomorphisms (of dg operads) connecting  $\tilde{\Lambda}$  (resp.  $f\tilde{\Lambda}$ ) and  $C_*(\mathcal{D})$  (resp.  $C_*(f\mathcal{D})$ ), which is the singular chain operad of  $\mathcal{D}$  (resp.  $f\mathcal{D}$ ).

**Remark 1.7.** Since  $J$  is a map of dg  $\tilde{\Lambda}$ -algebras, and  $H_*(\tilde{\Lambda}) \cong \mathcal{G}$ , (iii)-(c) shows that the map (2) preserves the Gerstenhaber structures.

**Remark 1.8.** As (iii)-(c) suggests, the chain complex  $C_*^{\mathcal{L}M}$  is a “geometric analogue” of the Hochschild complex  $C^*(\mathcal{A}_M, \mathcal{A}_M)$ . An important difference is that the isomorphism  $\mathbb{H}_*(\mathcal{L}M) \cong H_*(C_*^{\mathcal{L}M})$  holds for any closed oriented  $M$ , while we need some assumptions (e.g.  $\pi_1 M = 1$ ) to show that the map (2) is an isomorphism.



1.5.2. *Chain level loop product and bracket.* As the next proposition shows, one can define chain level loop product and bracket on  $C_*^{\mathcal{L}M}$ .

**Proposition 1.9.** *The loop product  $\bullet$  and the loop bracket  $\{, \}$  on  $\mathbb{H}_*(\mathcal{L}M)$  lift to operators on  $C_*^{\mathcal{L}M}$ , denoted by the same symbols.  $\bullet$  defines a dga algebra structure,  $\{, \}$  defines a dg Lie algebra structure on  $C_*^{\mathcal{L}M}$ , and the chain map  $J_M : C_*^{\mathcal{L}M} \rightarrow C^*(\mathcal{A}_M, \mathcal{A}_M)$  preserves these structures. Moreover, there exists an injective chain map  $\iota_M : (\mathcal{A}_M)_* \rightarrow C_*^{\mathcal{L}M}$  such that*

$$(3) \quad \iota_M(x) \bullet \iota_M(y) = \iota_M(x \wedge y), \quad \{\iota_M(x), \iota_M(y)\} = 0 \quad (\forall x, y \in \mathcal{A}_M)$$

and the following diagram commutes:

$$(4) \quad \begin{array}{ccc} \mathbb{H}_*(M) & \xrightarrow{\mathbb{H}_*(i_M)} & \mathbb{H}_*(\mathcal{L}M) \\ \cong \downarrow & & \downarrow \cong \\ H_{\text{dR}}^-(M) & \xrightarrow{H_*(\iota_M)} & H_*(C_*^{\mathcal{L}M}). \end{array}$$

As a consequence, one can define  $A_\infty$ -refinements of the loop product. Namely, by the homotopy transfer theorem (see Section 10.3 [24]), one can define an  $A_\infty$ -structure  $(\mu_k)_{k \geq 1}$  on  $\mathbb{H}_*(\mathcal{L}M)$ , such that  $\mu_1 = 0$ ,  $\mu_2 = \bullet$  and  $(\mathbb{H}_*(\mathcal{L}M), (\mu_k)_{k \geq 1})$  is homotopy equivalent to the dga algebra  $(C_*^{\mathcal{L}M}, \bullet)$ . Moreover, we may take  $(\mu_k)_{k \geq 1}$  so that  $\mu_k(\mathbb{H}_*(M)^{\otimes k}) \subset \mathbb{H}_*(M)$  for every  $k \geq 1$ , and  $(\mathbb{H}_*(M), (\mu_k)_{k \geq 1})$  is homotopy equivalent to the dga algebra  $(\mathcal{A}_M, d, \wedge)$ . In particular,  $(\mu_k)_{k \geq 1}$  recovers the classical Massey product on  $\mathbb{H}_*(M)$ .

By same arguments, one can define an  $L_\infty$ -structure  $(l_k)_{k \geq 1}$  on  $\mathbb{H}_*(\mathcal{L}M)$ , such that  $l_1 = 0$ ,  $l_2 = \{, \}$ , and  $(\mathbb{H}_*(\mathcal{L}M), (l_k)_{k \geq 1})$  is homotopy equivalent to the dg Lie algebra  $(C_*^{\mathcal{L}M}, \{, \})$ . Moreover, we may take  $(l_k)_{k \geq 1}$  so that  $l_k|_{\mathbb{H}_*(M)^{\otimes k}} = 0$  for any  $k \geq 1$ .

1.5.3. *Length filtration.* When  $M$  has a Riemannian metric,  $\mathbb{H}_*(\mathcal{L}M)$  is equipped with the length filtration, and it is shown in [18] that the loop product preserves this filtration. One of features of our chain model  $C_*^{\mathcal{L}M}$  is that, one can define the length filtration on this chain model so that the filtration is preserved by string topology operations.

For any  $\gamma \in \mathcal{L}M$ , let us denote  $\text{len}(\gamma) := \int_{S^1} |\dot{\gamma}|$ . For any  $a \in (0, \infty]$ , we define  $\mathcal{L}^a M := \{\gamma \in \mathcal{L}M \mid \text{len}(\gamma) < a\}$ . In general, a filtration (indexed by  $(0, \infty]$ ) on a chain complex  $C_*$  is a family  $(F^a C_*)_{a \in (0, \infty]}$  of subcomplexes of  $C_*$ , such that  $a \leq b \implies F^a C_* \subset F^b C_*$ . For any  $x \in C_*$ , we set  $|x| := \inf\{a \mid x \in F^a C_*\}$ .

**Proposition 1.10.** *Let  $M$  be a closed, oriented Riemannian manifold. Then, the chain complex  $C_*^{\mathcal{L}M}$  has a filtration  $(F^a C_*^{\mathcal{L}M})_{a \in (0, \infty]}$  such that the following conditions hold:*

- (i): *There exists an isomorphism  $\mathbb{H}_*(\mathcal{L}^a M) \cong H_*(F^a C_*^{\mathcal{L}M})$  for every  $a \in (0, \infty]$ , such that the following diagram is commutative for any  $a \leq b$ , where vertical maps are*

induced by inclusions:

$$\begin{array}{ccc} \mathbb{H}_*(\mathcal{L}^a M) & \xrightarrow{\cong} & H_*(F^a C^{\mathcal{L}^M}) \\ \downarrow & & \downarrow \\ \mathbb{H}_*(\mathcal{L}^b M) & \xrightarrow{\cong} & H_*(F^b C^{\mathcal{L}^M}). \end{array}$$

(ii): The  $f\tilde{\Lambda}$ -algebra structure on  $C_*^{\mathcal{L}^M}$  preserves the filtration. Namely,

$$|x \cdot (y_1 \otimes \cdots \otimes y_r)| \leq |y_1| + \cdots + |y_r| \quad (\forall x \in f\tilde{\Lambda}(r), \forall y_1, \dots, y_r \in C^{\mathcal{L}^M}).$$

**1.6. Previous works.** Rich algebraic structures in chain level string topology were outlined in [28] by D. Sullivan, and there have been several papers working out details.

X. Chen [8] introduced a chain model of the free loop space using Whitney differential forms, and defined several string topology operations (the loop product, loop bracket and rotation) on that chain model, recovering the BV-algebra structure on homology level ([8] also studies the  $S^1$ -equivariant case). It is not clear whether these operations extend to actions of a dg operad on this chain model.

On the other hand, a recent paper [13] by G. Drummond-Cole, K. Poirier and N. Rounds proposed a more geometric approach using short geodesic segments and diffuse intersection classes. [13] defines operations on the singular chain complex of the free loop space, recovering the homology level structure defined by Cohen-Godin [11]. In particular, [13] covers operations with multiple outputs and those corresponding to surfaces of higher genus. However, the operations in [13] are associative only up to homotopy, and it seems that the resulting algebraic structure is yet to be fully worked out.

**1.7. Potential applications to symplectic topology.** Let us discuss some potential applications of results in this paper to symplectic topology. For any  $C^\infty$ -manifold  $M$ , the cotangent bundle  $T^*M$  has the natural symplectic structure. When  $M$  is closed, oriented and spin, the Floer homology  $\mathrm{HF}_*(T^*M)$  has a BV algebra structure, and there exists an isomorphism of BV algebras  $\mathrm{HF}_*(T^*M) \cong \mathbb{H}_*(\mathcal{L}M)$  (see [1] and the references therein). There should be chain level refinements of this correspondence, and we expect that our chain level structures in string topology fit into this picture. More specifically, we expect that one can define an  $A_\infty$  (resp.  $L_\infty$ ) structure on  $\mathrm{HF}_*(T^*M)$  via counting solutions of appropriate Floer equations, which is homotopy equivalent to the  $A_\infty$  (resp.  $L_\infty$ ) refinement of the loop product (resp. bracket) on  $\mathbb{H}_*(\mathcal{L}M)$  defined in Section 1.5.2.

On the other hand, Fukaya [14] used chain level loop bracket for compactifications of the moduli space of pseudo-holomorphic disks with Lagrangian boundary conditions, and obtained restrictions on topological types of Lagrangian submanifolds. We expect that our definition of chain level loop bracket could be used to work out details of this approach.

Finally, there is a very interesting program by Cieliebak-Latschev [9], which compares symplectic field theory of sphere cotangent bundles and string topology of  $S^1$ -equivariant chains on free loops modulo constant loops. We hope that the construction in this paper will be a first step towards working out details of the string topology side of this program.

**1.8. Plan of this paper.** The rest of this paper is split into two parts. The goal of the first part (Sections 2–6) is to define the chain complex  $C_*^{\mathcal{L}^M}$ . This is a chain model of the free loop space  $\mathcal{L}M$ , carefully designed to define string topology operations on it. Our construction of the chain complex  $C_*^{\mathcal{L}^M}$  is based on two ideas.

The first idea is to introduce a notion of *de Rham chains*, which is a certain hybrid of singular chains and differential forms. We also need a notion of *differentiable spaces*, on which de Rham chains are defined. These notions are introduced in Section 2. We also define de Rham chain complex (the chain complex which consists of de Rham chains) for any differentiable space. An important step in our argument is to show that homology of the de Rham chain complex is naturally isomorphic to the usual singular homology. We prove it for finite-dimensional manifolds in Section 3, and for the free loop space in Section 4.

The second idea is to use (Moore) loops with marked points, which is inspired by the theory of iterated integrals. In Section 5, we introduce a space of Moore loops with marked points, and show that the collection of de Rham chain complexes of these spaces has a natural structure of a dg operad. Moreover, this dg operad is cyclic and has a multiplication and a unit. Based on results in Section 5, we define the chain complex  $C_*^{\mathcal{L}^M}$  in Section 6. We also prove results presented in Section 1.5 assuming Theorem 6.7, which is a purely algebraic result on operads.

The second part (Sections 7–11) is devoted to the proof of Theorem 6.7.

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## 2. DIFFERENTIABLE SPACES AND DE RHAM CHAINS

In this section, we introduce notions of *differentiable spaces* and *de Rham chain complexes*, which are basic for the arguments in this paper. The notion of differentiable spaces is introduced in Section 2.2., and de Rham chain complexes for these spaces are defined in Section 2.3. The rest of this section (2.4–2.8) is devoted to establishing several basic results about de Rham chain complexes of differentiable spaces.

**2.1. Integration along fibers.** Let  $P$  be a  $d$ -dimensional  $C^\infty$ -manifold. Throughout this paper, all manifolds are without boundary, unless otherwise specified. Let  $k \geq 0$  be an integer, and  $E \rightarrow P$  be an  $\mathbb{R}^k$ -bundle. We define  $\det E := \wedge^k E$ . When  $k = 0$ , we define  $\det E$  to be the trivial  $\mathbb{R}$ -bundle on  $P$ . Any exact sequence  $0 \rightarrow E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow 0$  of real vector bundles induces an isomorphism  $\det E_1 \cong \det E_2 \otimes \det E_0$ . An orientation of  $E$  is a section of the double cover  $(\det E)/\mathbb{R}_{>0} \rightarrow P$ . An orientation of  $TP \rightarrow P$  is called an orientation of  $P$ .

Recall that we denote  $\mathcal{A}^j(P) := C^\infty(\wedge^j T^*P)$  when  $j = 0, \dots, d$ . Let  $\mathcal{A}_c^j(P)$  denote its subspace which consists of compactly supported  $j$ -forms. When  $j \notin \{0, \dots, d\}$ , we set

$\mathcal{A}_c^j(P) = \mathcal{A}^j(P) = 0$ . When  $P$  is oriented, we define

$$\int_P : \mathcal{A}_c^*(P) \rightarrow \mathbb{R}; \quad \omega \mapsto \begin{cases} \int_P \omega & (* = d) \\ 0 & (* \neq d). \end{cases}$$

Let  $P_1, P_0$  be oriented  $C^\infty$ -manifolds, and  $\pi : P_1 \rightarrow P_0$  be a submersion (i.e.  $\pi$  is of class  $C^\infty$  and  $d\pi_p : T_p P_1 \rightarrow T_{\pi(p)} P_0$  is surjective for any  $p \in P_1$ ). Let  $d := \dim P_1 - \dim P_0$ . The *integration along fibers* is a chain map  $\pi_! : \mathcal{A}_c^*(P_1) \rightarrow \mathcal{A}_c^{*-d}(P_0)$ , which is defined in the following way.

First we consider the case  $P_1 = \mathbb{R}^{n+d}, P_0 = \mathbb{R}^n$  and  $\pi(x_1, \dots, x_n, y_1, \dots, y_d) = (x_1, \dots, x_n)$ . We assume that  $dx_1 \wedge \dots \wedge dx_n \wedge dy_1 \wedge \dots \wedge dy_d \in \mathcal{A}^{n+d}(\mathbb{R}^{n+d})$  and  $dx_1 \wedge \dots \wedge dx_n \in \mathcal{A}^n(\mathbb{R}^n)$  are positive with respect to orientations on  $\mathbb{R}^{n+d}$  and  $\mathbb{R}^n$ .

For  $\omega(x, y) := u(x, y) dx_{i_1} \dots dx_{i_k} dy_{j_1} \dots dy_{j_l} \in \mathcal{A}_c^*(\mathbb{R}^{n+d})$  where  $1 \leq i_1 < \dots < i_k \leq n$  and  $1 \leq j_1 < \dots < j_l \leq d$ , we define  $\pi_! \omega \in \mathcal{A}_c^{*-d}(\mathbb{R}^n)$  by

$$\pi_! \omega(x) := \begin{cases} 0 & (l < d) \\ \left( \int_{\mathbb{R}^d} u(x, y) dy_1 \dots dy_d \right) dx_{i_1} \dots dx_{i_k} & (l = d). \end{cases}$$

In the general case,  $\pi_!$  is defined by taking local charts and partitions of unity on  $P_1$ . Below is a list of some basic properties of the integration along fibers.

- If  $P_0$  is a positively oriented point, then  $\pi_! \omega = \int_{P_1} \omega$ .
- $\pi_!$  is a chain map, i.e. there holds  $d(\pi_! \omega) = \pi_!(d\omega)$ .
- For any  $\eta \in \mathcal{A}^*(P_0)$ ,  $\pi_!(\pi^* \eta \wedge \omega) = \eta \wedge \pi_! \omega$ .
- For any submersion  $\pi' : P_2 \rightarrow P_1$  and  $\omega \in \mathcal{A}_c^*(P_2)$ , there holds  $(\pi \circ \pi')_! \omega = \pi_!(\pi'_! \omega)$ .

**2.2. Differentiable spaces.** For any integers  $n \geq m \geq 0$ , let  $\mathcal{U}_{n,m}$  denote the set of oriented  $m$ -dimensional submanifolds in  $\mathbb{R}^n$ . We set  $\mathcal{U} := \bigsqcup_{n \geq m \geq 0} \mathcal{U}_{n,m}$ .

Let  $X$  be a set. A *differentiable structure* on  $X$  is a family of maps called *plots*, which satisfies the following conditions:

- Every plot is a map from  $U \in \mathcal{U}$  to  $X$ .
- If  $\varphi : U \rightarrow X$  is a plot,  $U' \in \mathcal{U}$  and  $\theta : U' \rightarrow U$  is a submersion, then  $\varphi \circ \theta : U' \rightarrow X$  is a plot.

A *differentiable space* is a pair of a set and a differentiable structure on it. For any differentiable space  $X$ , let  $\mathcal{P}(X) := \{(U, \varphi) \mid U \in \mathcal{U}, \varphi : U \rightarrow X \text{ is a plot}\}$ . A map  $f : X \rightarrow Y$  between differentiable spaces  $X$  and  $Y$  is called *smooth*, if there holds

$$(U, \varphi) \in \mathcal{P}(X) \implies (U, f \circ \varphi) \in \mathcal{P}(Y).$$

**Remark 2.1.** The term “plot” is originally used in the theory of Chen’s differentiable spaces ([7]) and the theory of diffeological spaces ([27], [19]). Our notion of differentiable spaces is weaker than these notions. In particular, in axioms of both of these spaces, it

is required that all constant maps are plots, while we do not require this condition (see Example 2.2 (i)-(b) below).

**Example 2.2.** Let us explain some examples of differentiable structures.

(i): Let  $M$  be a  $C^\infty$ -manifold. One can consider the following differentiable structures on  $M$ :

(a):  $\varphi : U \rightarrow M$  is a plot if  $\varphi$  is of class  $C^\infty$ . We denote the resulting differentiable space as  $M$ .

(b):  $\varphi : U \rightarrow M$  is a plot if  $\varphi$  is a submersion (we always assume that any submersion is of class  $C^\infty$ ). We denote the resulting differentiable space as  $M_{\text{reg}}$ .

The identity map  $\text{id}_M : M_{\text{reg}} \rightarrow M$  is smooth, but  $\text{id}_M : M \rightarrow M_{\text{reg}}$  is not.

(ii):  $\mathcal{L}M := C^\infty(S^1, M)$  has the following differentiable structure: a map  $\varphi : U \rightarrow \mathcal{L}M$  is a plot if  $U \times S^1 \rightarrow M; (u, \theta) \mapsto \varphi(u)(\theta)$  is of class  $C^\infty$ .

(iii): Let  $X$  be a differentiable space,  $Y$  a subset of  $X$ , and  $i : Y \rightarrow X$  be the inclusion map. One can define the following differentiable structure on  $Y$ : a map  $\varphi : U \rightarrow Y$  is a plot if  $i \circ \varphi : U \rightarrow X$  is a plot of  $X$ .

(iv): Let  $(X_s)_{s \in S}$  be a family of differentiable spaces parametrized by the nonempty set  $S$ . The product  $X := \prod_{s \in S} X_s$  has the following differentiable structure:  $\varphi : U \rightarrow X$  is a plot if  $\pi_s \circ \varphi$  is a plot of  $X_s$  for any  $s \in S$  ( $\pi_s$  denotes the projection to  $X_s$ ).

Two smooth maps  $f, g : X \rightarrow Y$  are called *smoothly homotopic*, if there exists a smooth map  $h : X \times \mathbb{R} \rightarrow Y$  such that

$$h(x, s) = \begin{cases} f(x) & (s < 0), \\ g(x) & (s > 1). \end{cases}$$

$h$  is called a *smooth homotopy* between  $f$  and  $g$ .

**Remark 2.3.** The differentiable structure on  $\mathbb{R}$  is defined as in Example 2.2 (i)-(a), i.e.  $(U, \varphi) \in \mathcal{P}(\mathbb{R}) \iff \varphi \in C^\infty(U, \mathbb{R})$ . The differentiable structure on  $X \times \mathbb{R}$  is defined as in Example 2.2 (iv).

**Remark 2.4.** It seems that transitivity of  $\sim$  does not hold in general.

Unless otherwise specified, any  $C^\infty$ -manifold  $M$  will be equipped with the differentiable structure in Example 2.2 (i)-(a). When  $C^\infty$ -manifolds  $M, N$  are equipped with these differentiable structures, a map  $f : M \rightarrow N$  is smooth if and only if  $f$  is of class  $C^\infty$ .

**2.3. de Rham chain complex.** Let  $X$  be a differentiable space. For any  $k \in \mathbb{Z}$ , we set

$$\bar{C}_k^{\text{dR}}(X) := \bigoplus_{(U, \varphi) \in \mathcal{P}(X)} \mathcal{A}_c^{\dim U - k}(U).$$

Notice that  $\bar{C}_k^{\text{dR}}(X) = 0$  for any  $k < 0$ .

For any  $(U, \varphi) \in \mathcal{P}(X)$  and  $\omega \in \mathcal{A}_c^{\dim U - k}(U)$ , let  $(U, \varphi, \omega)$  denote the image of  $\omega$  by the natural injection  $\mathcal{A}_c^{\dim U - k}(U) \rightarrow \bar{C}_k^{\text{dR}}(X)$ . Let  $Z_k(X)$  denote the subspace of  $\bar{C}_k^{\text{dR}}(X)$ ,

which is generated by

$$\{(U, \varphi, \pi_! \omega) - (V, \varphi \circ \pi, \omega) \mid (U, \varphi) \in \mathcal{P}(X), \quad V \in \mathcal{U}, \quad \omega \in \mathcal{A}_c^{\dim V - k}(V), \\ \pi : V \rightarrow U \text{ is a submersion}\}.$$

We define  $C_k^{\text{dR}}(X) := \bar{C}_k^{\text{dR}}(X)/Z_k(X)$ . For every  $k \in \mathbb{Z}$ ,

$$\partial : C_k^{\text{dR}}(X) \rightarrow C_{k-1}^{\text{dR}}(X); \quad [(U, \varphi, \omega)] \mapsto [(U, \varphi, d\omega)]$$

is well-defined, since  $d(\pi_! \omega) = \pi_!(d\omega)$ . Moreover,  $\partial^2 = 0$  since  $d^2 = 0$ . We call  $(C_*^{\text{dR}}(X), \partial)$  the *de Rham chain complex* of  $X$ , and denote its homology as  $H_*^{\text{dR}}(X)$ . Elements of  $C_*^{\text{dR}}(X)$  are called *de Rham chains* of  $X$ .

**Remark 2.5.** Our notion of de Rham chains is inspired by the notion of *approximate de Rham chains* by K. Fukaya ([14] Definition 6.4). However, an explicit definition of a chain complex is not given in [14].

The *augmentation map*  $\varepsilon : C_0^{\text{dR}}(X) \rightarrow \mathbb{R}$  is defined as

$$\varepsilon([(U, \varphi, \omega)]) := \int_U \omega.$$

$\varepsilon$  vanishes on  $\partial C_1^{\text{dR}}(X)$  by Stokes' theorem.

Next we define the *fiber product* on de Rham chain complexes. Let  $M$  be an oriented  $C^\infty$ -manifold of dimension  $d$ . Let us consider the differentiable space  $M_{\text{reg}}$  in Example 2.2 (i)-(b). Let  $X, Y$  be differentiable spaces, and  $e_X : X \rightarrow M_{\text{reg}}, e_Y : Y \rightarrow M_{\text{reg}}$  be smooth maps. We define a differentiable structure on

$$X \times_M Y := \{(x, y) \in X \times Y \mid e_X(x) = e_Y(y)\}$$

as a subset of  $X \times Y$  (see Example 2.2 (iii) and (iv)).

We are going to define a chain map

$$(5) \quad C_{k+d}^{\text{dR}}(X) \otimes C_{l+d}^{\text{dR}}(Y) \rightarrow C_{k+l+d}^{\text{dR}}(X \times_M Y); \quad a \otimes b \mapsto a \times_M b,$$

which we call the fiber product on de Rham chain complexes.

Let  $(U, \varphi) \in \mathcal{P}(X), (V, \psi) \in \mathcal{P}(Y)$ . Then,  $e_U := e_X \circ \varphi : U \rightarrow M$  and  $e_V := e_Y \circ \psi : V \rightarrow M$  are submersions. Thus,  $U \times_M V := \{(u, v) \in U \times V \mid e_U(u) = e_V(v)\}$  is a  $C^\infty$ -manifold, moreover it is a submanifold of the Euclidean space (since  $U, V \in \mathcal{U}$ ). The map  $e_{UV} : U \times_M V \rightarrow M; (u, v) \mapsto e_U(u) = e_V(v)$  is also a submersion.

Let us define an orientation on  $U \times_M V$ . Let  $F_U := \text{Ker } de_U \subset TU, F_V := \text{Ker } de_V \subset TV$ . There exist exact sequences

$$0 \rightarrow F_U \rightarrow TU \rightarrow e_U^* TM \rightarrow 0, \quad 0 \rightarrow F_V \rightarrow TV \rightarrow e_V^* TM \rightarrow 0, \\ 0 \rightarrow F_U \oplus F_V \rightarrow T(U \times_M V) \rightarrow e_{UV}^* TM \rightarrow 0.$$

Then we obtain isomorphisms

$$\det(TU) \cong (e_U)^* \det(TM) \otimes \det(F_U), \\ \det(TV) \cong (e_V)^* \det(TM) \otimes \det(F_V), \\ \det(T(U \times_M V)) \cong (e_{UV})^* \det(TM) \otimes \det(F_U) \otimes \det(F_V).$$

Since  $M, U, V$  are oriented, one can define an orientation on  $F_U$  (resp.  $F_V$ ) so that the first (resp. second) isomorphism is orientation preserving. Then, one can define an orientation of  $U \times_M V$  so that the third isomorphism is orientation preserving. Now,  $U \times_M V$  is an oriented submanifold of the Euclidean space. Therefore,  $U \times_M V$  is an element of  $\mathcal{U}$ .

Let  $\pi_X : X \times_M Y \rightarrow X$ ,  $\pi_Y : X \times_M Y \rightarrow Y$ ,  $\pi_U : U \times_M V \rightarrow U$ ,  $\pi_V : U \times_M V \rightarrow V$  be projection maps. Then

- Since  $\pi_U$  is a submersion,  $(U \times_M V, \pi_X \circ (\varphi \times \psi)) = (U \times_M V, \varphi \circ \pi_U) \in \mathcal{P}(X)$ .
- Since  $\pi_V$  is a submersion,  $(U \times_M V, \pi_Y \circ (\varphi \times \psi)) = (U \times_M V, \psi \circ \pi_V) \in \mathcal{P}(Y)$ .

Therefore  $(U \times_M V, \varphi \times \psi) \in \mathcal{P}(X \times_M Y)$ . Now, let us define (5) by

$$[(U, \varphi, \omega)] \times_M [(V, \psi, \eta)] := (-1)^{l(\dim U - d)} [(U \times_M V, \varphi \times \psi, \omega \times \eta)].$$

By direct computations, one can check that this is a well-defined chain map. The product is associative, and a smooth map  $f : X \times_M Y \rightarrow Y \times_M X; (x, y) \mapsto (y, x)$  satisfies  $f_*(a \times_M b) = (-1)^{kl} b \times_M a$ .

When  $M$  is a positively oriented point (thus  $d = 0$ ), the chain map (5) is

$$C_k^{\text{dR}}(X) \otimes C_l^{\text{dR}}(Y) \rightarrow C_{k+l}^{\text{dR}}(X \times Y); \quad a \otimes b \mapsto a \times b,$$

which we call the *cross product* on de Rham chain complexes.

**2.4. de Rham chain complex of pt.** Let  $\text{pt}$  be a set which has a unique element. For any  $U \in \mathcal{U}$ , there exists a unique map  $\varphi_U : U \rightarrow \text{pt}$ . We define a differentiable structure on  $\text{pt}$  by  $\mathcal{P}(\text{pt}) := \{(U, \varphi_U) \mid U \in \mathcal{U}\}$ .

Let us consider  $\{0\} \in \mathcal{U}_{1,0} \subset \mathcal{U}$  with the positive orientation. For any  $U \in \mathcal{U}$ , let us denote the unique map  $U \rightarrow \{0\}$  by  $\pi_U$ . For any  $\omega \in \mathcal{A}_c^*(U)$ , we obtain

$$[(U, \varphi_U, \omega)] = [(\{0\}, \varphi_{\{0\}}, (\pi_U)_! \omega)] = [(\{0\}, \varphi_{\{0\}}, \int_U \omega)].$$

Therefore,  $C_k^{\text{dR}}(\text{pt}) = 0$  if  $k \neq 0$ , and the augmentation map  $\varepsilon : C_0^{\text{dR}}(U) \rightarrow \mathbb{R}$  is an isomorphism. In particular,

$$H_k^{\text{dR}}(\text{pt}) \cong \begin{cases} \mathbb{R} & (k = 0), \\ 0 & (k \neq 0). \end{cases}$$

**2.5. Functoriality.** For any differentiable spaces  $X, Y$  and a smooth map  $f : X \rightarrow Y$ ,

$$f_* : C_*^{\text{dR}}(X) \rightarrow C_*^{\text{dR}}(Y); \quad [(U, \varphi, \omega)] \mapsto [(U, f \circ \varphi, \omega)]$$

is a well-defined chain map.

**Proposition 2.6.** *Let  $X, Y$  be differentiable spaces, and  $f, g : X \rightarrow Y$  be smooth maps. If  $f, g$  are smoothly homotopic, then  $f_*, g_* : C_*^{\text{dR}}(X) \rightarrow C_*^{\text{dR}}(Y)$  are chain homotopic.*

**Proof.** Let  $h : X \times \mathbb{R} \rightarrow Y$  be a smooth homotopy between  $f$  and  $g$ , i.e.,  $h$  is a smooth map such that  $h(x, s) = f(x)$  if  $s < 0$  and  $h(x, s) = g(x)$  if  $s > 1$ .

Take  $a \in C_c^\infty(\mathbb{R})$  so that  $a \equiv 1$  on  $[-\varepsilon, 1 + \varepsilon]$  for some  $\varepsilon > 0$ . Let  $u := [(\mathbb{R}, \text{id}_{\mathbb{R}}, a)] \in C_1^{\text{dR}}(\mathbb{R})$ , and define a linear map  $K : C_*^{\text{dR}}(X) \rightarrow C_{*+1}^{\text{dR}}(Y)$  by  $K(x) := (-1)^{|x|} h_*(x \times u)$ . Since  $h_*$  and the cross product are chain maps, for any  $x \in C_*^{\text{dR}}(X)$ , we have

$$\partial K(x) + K(\partial x) = (-1)^{|x|} h_*(\partial(x \times u)) + (-1)^{|x|-1} h_*(\partial x \times u) = h_*(x \times \partial u).$$

Thus, it is enough to show that  $h_*(x \times \partial u) = f_*(x) - g_*(x)$ . Since both sides are linear on  $x$ , we may assume that  $x = [(U, \varphi, \omega)]$  for some  $(U, \varphi) \in \mathcal{P}(X)$  and  $\omega \in \mathcal{A}_c^*(U)$ .

Let  $i_0 : \mathbb{R}_{<0} \rightarrow \mathbb{R}$ ,  $i_1 : \mathbb{R}_{>1} \rightarrow \mathbb{R}$  be the inclusion maps, and  $\alpha_0 := da|_{\mathbb{R}_{<0}}$ ,  $\alpha_1 := da|_{\mathbb{R}_{>1}}$ . Define  $v_0, v_1 \in C_0^{\text{dR}}(\mathbb{R})$  by  $v_0 := [(\mathbb{R}_{<0}, i_0, \alpha_0)]$ ,  $v_1 := [(\mathbb{R}_{>1}, i_1, \alpha_1)]$ . Then, since  $da$  is supported on  $\mathbb{R}_{<0} \cup \mathbb{R}_{>1}$ , we have  $\partial u = v_0 + v_1$ .

Let  $x = [(U, \varphi, \omega)] \in C_*^{\text{dR}}(X)$ . Then,

$$h_*(x \times v_0) = [(U \times \mathbb{R}_{<0}, h \circ (\varphi \times i_0), \omega \times \alpha_0)].$$

There holds  $h \circ (\varphi \times i_0) = f \circ \varphi \circ \text{pr}_U$ , where  $\text{pr}_U : U \times \mathbb{R}_{<0} \rightarrow U$  is the projection map. Moreover,  $(\text{pr}_U)_!(\omega \times \alpha_0) = \omega$ , since  $\int_{\mathbb{R}_{<0}} \alpha_0 = a(0) = 1$ . Thus,  $h_*(x \times v_0) = f_*(x)$ . A similar argument shows  $h_*(x \times v_1) = -g_*(x)$ . Hence, we get

$$h_*(x \times \partial u) = h_*(x \times (v_0 + v_1)) = f_*(x) - g_*(x).$$

□

**2.6. Truncated spaces.** Let  $X$  be a differentiable space. For any function  $f : X \rightarrow \mathbb{R}$  and  $a \in \mathbb{R}$ , let us define

$$X_{f,a} := \{x \in X \mid f(x) < a\}.$$

This is a subspace of  $X$  truncated by the inequality  $f(x) < a$ . We prove some technical results about de Rham chain complexes of these truncated spaces.

A function  $f : X \rightarrow \mathbb{R}$  is called smooth, if  $f \circ \varphi \in C^\infty(U)$  for any  $(U, \varphi) \in \mathcal{P}(X)$ .  $f$  is called *approximately smooth*, if there exists a decreasing sequence  $(f_j)_{j \geq 1}$  of smooth functions on  $X$  such that  $f(x) = \lim_{j \rightarrow \infty} f_j(x)$  for any  $x \in X$ .

**Remark 2.7.** If  $f : X \rightarrow \mathbb{R}$  is approximately smooth,  $f \circ \varphi : U \rightarrow \mathbb{R}$  is upper semi-continuous for any  $(U, \varphi) \in \mathcal{P}(X)$ . This is because  $(f \circ \varphi)^{-1}(\mathbb{R}_{<a}) = \bigcup_{j=1}^{\infty} (f_j \circ \varphi)^{-1}(\mathbb{R}_{<a})$  for any  $a \in \mathbb{R}$ .

An important example of an approximately smooth (but not smooth) function is the length functional (see Section 1.5.3) on the free loop space.

**Lemma 2.8.** *Let  $M$  be a Riemannian manifold, and consider the differentiable structure on  $\mathcal{L}M$  as in Example 2.2 (ii). Then,  $\text{len} : \mathcal{L}M \rightarrow \mathbb{R}$  is approximately smooth.*

**Proof.** It is easy to check that, for any  $\rho \in C^\infty(\mathbb{R}_{\geq 0})$ , a functional

$$\mathcal{E}_\rho : \mathcal{L}M \rightarrow \mathbb{R}; \quad \gamma \mapsto \int_{S^1} \rho(|\dot{\gamma}|^2)$$



is smooth. Let us take a decreasing sequence  $(\rho_j)_{j \geq 1}$  on  $C^\infty(\mathbb{R}_{\geq 0})$ , such that  $\lim_{j \rightarrow \infty} \rho_j(t) = \sqrt{t}$  for any  $t \geq 0$ . Then,  $(\mathcal{E}_{\rho_j})_{j \geq 1}$  is a decreasing sequence of smooth functions on  $\mathcal{L}M$ , such that  $\lim_{j \rightarrow \infty} \mathcal{E}_{\rho_j} = \text{len}$ .  $\square$

**Lemma 2.9.** *Let  $(f_j)_{j \geq 1}$  be a decreasing sequence of approximately smooth functions on  $X$ , such that  $\lim_{j \rightarrow \infty} f_j(x) < 0$  for any  $x \in X$ . Then, the chain map  $\varinjlim_{j \rightarrow \infty} C_*^{\text{dR}}(X_{f_j,0}) \rightarrow C_*^{\text{dR}}(X)$ , which is induced by the inclusion maps, is surjective.*

**Proof.** Let  $(U, \varphi) \in \mathcal{P}(X)$  and  $\omega \in \mathcal{A}_c^*(U)$ . Setting  $\varphi_j := f_j \circ \varphi$  for each  $j \geq 1$ ,  $(\varphi_j)_{j \geq 1}$  is a decreasing sequence of upper semi-continuous functions on  $U$ . Since  $\text{supp } \omega$  is compact, there exists  $j$  such that  $\varphi_j(u) < 0$  for any  $u \in \text{supp } \omega$ . Setting  $U_j := \{u \in U \mid \varphi_j(u) < 0\}$ , the chain map  $C_*^{\text{dR}}(X_{f_j,0}) \rightarrow C_*^{\text{dR}}(X)$  maps  $[(U_j, \varphi|_{U_j}, \omega|_{U_j})]$  to  $[(U, \varphi, \omega)]$ .  $\square$

**Lemma 2.10.** *Let  $f : X \rightarrow \mathbb{R}$  be an approximately smooth function. Then, the chain map  $I_0 : C_*^{\text{dR}}(X_{f,0}) \rightarrow C_*^{\text{dR}}(X)$ , which is induced by the inclusion map, is injective.*

**Proof.** First let us consider the case when  $f$  is a smooth function on  $X$ . For any  $c < 0$ ,

$$I_{c,0} : C_*^{\text{dR}}(X_{f,c}) \rightarrow C_*^{\text{dR}}(X_{f,0}), \quad I_c : C_*^{\text{dR}}(X_{f,c}) \rightarrow C_*^{\text{dR}}(X)$$

denote the chain maps induced by the inclusion maps. Suppose that  $u \in C_*^{\text{dR}}(X_{f,0})$  satisfies  $I_0(u) = 0$ . By the previous lemma, there exists  $c < 0$  such that  $u \in \text{Im } I_{c,0}$ . Take  $v \in C_*^{\text{dR}}(X_{f,c})$  so that  $u = I_{c,0}(v)$ .

If there exists a linear map  $J : C_*^{\text{dR}}(X) \rightarrow C_*^{\text{dR}}(X_{f,0})$  such that  $J \circ I_c = I_{c,0}$ , we can prove  $u = 0$  by

$$u = I_{c,0}(v) = J \circ I_c(v) = J \circ I_0 \circ I_{c,0}(v) = J \circ I_0(u) = 0.$$

To define such  $J$ , we fix  $\kappa \in C^\infty(\mathbb{R}, [0, 1])$  so that  $\kappa \equiv 1$  on  $\mathbb{R}_{\leq c}$  and  $\text{supp } \kappa \subset \mathbb{R}_{< 0}$ . For any  $(U, \varphi) \in \mathcal{P}(X)$ , we set  $U_{f,0} := \{u \in U \mid f(\varphi(u)) < 0\}$ . Then, a linear map

$$J : C_*^{\text{dR}}(X) \rightarrow C_*^{\text{dR}}(X_{f,0}); \quad [(U, \varphi, \omega)] \mapsto [(U_{f,0}, \varphi|_{U_{f,0}}, ((\kappa \circ f \circ \varphi) \cdot \omega)|_{U_{f,0}})]$$

is well-defined (it is *not* a chain map).  $J \circ I_c = I_{c,0}$  is obvious since  $\kappa \equiv 1$  on  $\mathbb{R}_{\leq c}$ . This completes the proof when  $f$  is a smooth function on  $X$ .

Finally, we consider the case when  $f$  is any approximately smooth function on  $X$ . By definition, there exists a decreasing sequence of smooth functions  $(f_j)_{j \geq 1}$  such that  $\lim_{j \rightarrow \infty} f_j = f$ . The previous lemma shows that the chain map  $\varinjlim_{j \rightarrow \infty} C_*^{\text{dR}}(X_{f_j,0}) \rightarrow C_*^{\text{dR}}(X_{f,0})$  is surjective. On the other hand, since each  $f_j$  is smooth,  $C_*^{\text{dR}}(X_{f_j,0}) \rightarrow C_*^{\text{dR}}(X)$  is injective. Thus  $C_*^{\text{dR}}(X_{f,0}) \rightarrow C_*^{\text{dR}}(X)$  is injective.  $\square$

**Corollary 2.11.** *Let  $(f_j)_{j \geq 1}$  be any decreasing sequence of approximately smooth functions on  $X$ , such that  $\lim_{j \rightarrow \infty} f_j(x) < 0$  for any  $x \in X$ . Then, the chain map  $\varinjlim_{j \rightarrow \infty} C_*^{\text{dR}}(X_{f_j,0}) \rightarrow C_*^{\text{dR}}(X)$ , which is induced by the inclusion maps, is an isomorphism.*

**Proof.** We already prove the surjectivity in Lemma 2.9. By Lemma 2.10,  $C_*^{\text{dR}}(X_{f_j,0}) \rightarrow C_*^{\text{dR}}(X)$  is injective for every  $j$ , thus  $\varinjlim_{j \rightarrow \infty} C_*^{\text{dR}}(X_{f_j,0}) \rightarrow C_*^{\text{dR}}(X)$  is also injective.  $\square$

**2.7. Smooth singular chains.** We define *smooth singular chains* on differentiable spaces, and compare them with de Rham chains.

Let  $X$  be a differentiable space, and  $k \geq 0$  be an integer. A map  $\sigma : \Delta^k \rightarrow X$  is called *strongly smooth*, if there exists an open neighborhood  $U$  of  $\Delta^k \subset \mathbb{R}^k$ , and a smooth map  $\bar{\sigma} : U \rightarrow X$  such that  $\bar{\sigma}|_{\Delta^k} = \sigma$ .  $\Delta^k$  and  $U$  are equipped with the differentiable structures as subsets of  $\mathbb{R}^k$ .

For any  $k \geq 0$ , let  $C_k^{\text{sm}}(X)$  denote the  $\mathbb{R}$ -vector space generated by all strongly smooth maps  $\Delta^k \rightarrow X$ . For any  $k < 0$ , we set  $C_k^{\text{sm}}(X) := 0$ . A differential on  $C_*^{\text{sm}}(X)$  is defined in the same way as in the singular chain complex. To fix notations, let us spell out details. For any  $k \geq 1$  and  $0 \leq j \leq k$ , we define a map  $d_{k,j} : \Delta^{k-1} \rightarrow \Delta^k$  by

$$(6) \quad d_{k,j}(t_1, \dots, t_{k-1}) := \begin{cases} (0, t_1, \dots, t_{k-1}) & (j = 0), \\ (t_1, \dots, t_j, t_j, \dots, t_{k-1}) & (1 \leq j \leq k-1), \\ (t_1, \dots, t_{k-1}, 1) & (j = k). \end{cases}$$

In particular,  $d_{1,j} : \Delta^0 \rightarrow \Delta^1$  is defined as  $d_{1,j}(0) = j$  for  $j = 0, 1$ . For any  $k \geq 1$ , a differential  $\partial : C_k^{\text{sm}}(X) \rightarrow C_{k-1}^{\text{sm}}(X)$  is defined as

$$\partial\sigma := \sum_{j=0}^k (-1)^j \sigma \circ d_{k,j}.$$

We call the chain complex  $(C_*^{\text{sm}}(X), \partial)$  *smooth singular chain complex*, and its homology  $H_*^{\text{sm}}(X)$  *smooth singular homology*. For any smooth map  $f : X \rightarrow Y$  between differentiable spaces, one can define the chain map  $f_* : C_*^{\text{sm}}(X) \rightarrow C_*^{\text{sm}}(Y)$  in the obvious way. If smooth maps  $f, g : X \rightarrow Y$  are smoothly homotopic,  $f_*, g_* : C_*^{\text{sm}}(X) \rightarrow C_*^{\text{sm}}(Y)$  are chain homotopic.

For any  $k \geq 1$ , we define  $\delta_k : C_*^{\text{dR}}(\Delta^{k-1}) \rightarrow C_*^{\text{dR}}(\Delta^k)$  by  $\delta_k := \sum_{j=0}^k (-1)^j (d_{k,j})_*$ . Then,  $\delta_{k+1} \circ \delta_k = 0$  for any  $k \geq 1$ .

**Lemma 2.12.** (i): *There exists a sequence  $(u_k)_{k \geq 0}$  such that  $u_k \in C_k^{\text{dR}}(\Delta^k)$  for any  $k \geq 0$ ,  $u_0 \in C_0^{\text{dR}}(\Delta^0)$  is characterized by  $\varepsilon(u_0) = 1$ , and*

$$\partial u_k = \delta_k(u_{k-1}) \quad (\forall k \geq 1).$$

(ii): *Suppose that  $(u_k)_{k \geq 0}$  and  $(u'_k)_{k \geq 0}$  satisfy the conditions in (i). Then there exists a sequence  $(v_k)_{k \geq 1}$  such that  $v_k \in C_{k+1}^{\text{dR}}(\Delta^k)$  for any  $k \geq 1$ ,  $\partial v_1 = u_1 - u'_1$ , and*

$$\partial v_k = u_k - u'_k - \delta_k(v_{k-1}) \quad (\forall k \geq 2).$$

**Proof.** For any  $k \geq 0$ ,  $\Delta^k$  is smoothly homotopic to pt, thus  $H_*^{\text{dR}}(\Delta^k) \cong H_*^{\text{dR}}(\text{pt})$ . Using this fact, the assertions are easy to prove by induction on  $k$ .  $\square$

For any  $u = (u_k)_{k \geq 0}$  which satisfies Lemma 2.12 (i), one can define a natural transformation  $\iota^u : C_*^{\text{sm}} \rightarrow C_*^{\text{dR}}$  as follows (we set  $\iota_k^u = 0$  when  $k < 0$ ):

$$\iota^u(X)_k : C_k^{\text{sm}}(X) \rightarrow C_k^{\text{dR}}(X); \quad \sigma \mapsto \sigma_*(u_k).$$

The equation  $\partial u_k = \delta_k(u_{k-1})$  shows that  $\iota^u(X)_*$  is a chain map. Lemma 2.12 (ii) shows that the homotopy equivalence class of  $\iota^u(X)_*$  does not depend on choices of  $u$ . In particular, the linear map  $H_*(\iota^u(X)) : H_*^{\text{sm}}(X) \rightarrow H_*^{\text{dR}}(X)$  does not depend on  $u$ .

Finally, we define the cross product on  $C_*^{\text{sm}}$ . Let us take  $\tau_{k,l} \in C_{k+l}^{\text{sm}}(\Delta^k \times \Delta^l)$  for all  $k, l \geq 0$ , such that  $\tau_{0,0}$  is characterized by  $\varepsilon(\tau_{0,0}) = 1$ , and the following equation holds for any  $k, l \geq 0$ :

$$\partial \tau_{k,l} = \sum_{0 \leq i \leq k} (-1)^i (d_{k,i} \times \text{id}_{\Delta^l})_*(\tau_{k-1,l}) + (-1)^k \sum_{0 \leq j \leq l} (-1)^j (\text{id}_{\Delta^k} \times d_{l,j})_*(\tau_{k,l-1}).$$

We define the cross product  $C_k^{\text{sm}}(X) \otimes C_l^{\text{sm}}(Y) \rightarrow C_{k+l}^{\text{sm}}(X \times Y)$  by

$$\sigma_X \otimes \sigma_Y \mapsto (\sigma_X \times \sigma_Y)_*(\tau_{k,l}) \quad (\sigma_X : \Delta^k \rightarrow X, \sigma_Y : \Delta^l \rightarrow Y, k, l \geq 0).$$

The homotopy equivalence class of this map does not depend on choices of  $(\tau_{k,l})_{k,l \geq 0}$ .

On the other hand, we defined the cross product for de Rham chains (see Section 2.3). The next lemma is proved by the standard method of acyclic models.

**Lemma 2.13.** *For any differentiable spaces  $X, Y$  and  $u = (u_k)_{k \geq 0}$  satisfying Lemma 2.12 (i), the following diagram commutes up to homotopy, where horizontal maps are cross products:*

$$\begin{array}{ccc} C_*^{\text{sm}}(X) \otimes C_*^{\text{sm}}(Y) & \longrightarrow & C_*^{\text{sm}}(X \times Y) \\ \downarrow \iota^u(X) \otimes \iota^u(Y) & & \downarrow \iota^u(X \times Y) \\ C_*^{\text{dR}}(X) \otimes C_*^{\text{dR}}(Y) & \longrightarrow & C_*^{\text{dR}}(X \times Y). \end{array}$$

**2.8. Integration over de Rham chains.** Let  $M$  be a  $C^\infty$ -manifold, and  $n \geq 0$ . We define  $\langle, \rangle : \mathcal{A}^n(M) \otimes C_n^{\text{dR}}(M) \rightarrow \mathbb{R}$  by

$$\langle \alpha, [(U, \varphi, \omega)] \rangle := \int_U \varphi^* \alpha \wedge \omega.$$

It induces a linear map  $H_{\text{dR}}^n(M) \otimes H_n^{\text{dR}}(M) \rightarrow \mathbb{R}$ , which we also denote by  $\langle, \rangle$ .

For any subset  $S \subset M$ , let  $\mathcal{A}^*(S) := \varinjlim_U \mathcal{A}^*(U)$  where  $U$  runs over all open neighborhoods of  $S$ , and let  $H_{\text{dR}}^*(S) := H^*(\mathcal{A}^*(S), d)$ . Then, one can define  $H_{\text{dR}}^*(S) \otimes H_*^{\text{dR}}(S) \rightarrow \mathbb{R}$ . In the next lemma, we consider the case  $S = \Delta^{k_1} \times \dots \times \Delta^{k_m} \subset \mathbb{R}^{k_1 + \dots + k_m}$ .

**Lemma 2.14.** *Suppose that  $(u_k)_{k \geq 0}$  satisfies Lemma 2.12 (i). For any nonnegative integers  $k = k_1 + \dots + k_m$  and  $\alpha \in \mathcal{A}^k(\Delta^{k_1} \times \dots \times \Delta^{k_m})$ ,*

$$\langle \alpha, u_{k_1} \times \dots \times u_{k_m} \rangle = (-1)^{k(k-1)/2} \int_{\Delta^{k_1} \times \dots \times \Delta^{k_m}} \alpha.$$

**Proof.** We fix  $m$  and prove the lemma by induction on  $k$ . When  $k = 0$ , i.e.  $k_1 = \dots = k_m = 0$ , the lemma can be directly checked. If the lemma is established for  $k \leq N - 1$ ,

the case  $k = N$  is proved as follows. Let us take  $\beta \in \mathcal{A}^{k-1}(\Delta^{k_1} \times \cdots \times \Delta^{k_m})$  such that  $d\beta = \alpha$  (this is always possible since  $H_{\text{dR}}^k(\Delta^{k_1} \times \cdots \times \Delta^{k_m}) = 0$ ). Then,

$$\begin{aligned} \langle d\beta, u_{k_1} \times \cdots \times u_{k_m} \rangle &= (-1)^k \sum_{j=1}^m (-1)^{k_1 + \cdots + k_{j-1}} \langle \beta, u_{k_1} \times \cdots \times \partial u_{k_j} \times \cdots \times u_{k_m} \rangle \\ &= (-1)^{k+(k-1)(k-2)/2} \int_{\partial(\Delta^{k_1} \times \cdots \times \Delta^{k_m})} \beta \\ &= (-1)^{k(k-1)/2} \int_{\Delta^{k_1} \times \cdots \times \Delta^{k_m}} d\beta. \end{aligned}$$

The second equality follows from the induction hypothesis, and the last equality follows from Stokes' theorem.  $\square$

### 3. DE RHAM CHAINS ON $C^\infty$ -MANIFOLDS

Let  $M$  be an oriented  $C^\infty$ -manifold, and consider the differentiable structure as in Example 2.2 (i)-(a), i.e.  $\mathcal{P}(M) := \{(U, \varphi) \mid U \in \mathcal{U}, \varphi \in C^\infty(U, M)\}$ . With this differentiable structure, a map  $\Delta^k \rightarrow M$  is strongly smooth if it extends to a  $C^\infty$ -map  $U \rightarrow M$  for some open neighborhood  $U$  of  $\Delta^k \subset \mathbb{R}^k$ . In particular,  $C_*^{\text{sm}}(M)$  is a subcomplex of  $C_*(M)$  (the usual singular chain complex of  $M$ ). It is known that  $C_*^{\text{sm}}(M) \rightarrow C_*(M)$  is a quasi-isomorphism (see [23] Theorem 18.7). Therefore, we obtain the natural isomorphism  $H_*^{\text{sm}}(M) \cong H_*(M)$ .

In Section 2.7, we defined the map  $H_*^{\text{sm}}(X) \rightarrow H_*^{\text{dR}}(X)$  for any differentiable space  $X$ . The goal of this section is to prove the following Theorem 3.1. As an immediate consequence, we obtain the natural isomorphism  $H_*(M) \cong H_*^{\text{dR}}(M)$ .

**Theorem 3.1.** *For any oriented  $C^\infty$ -manifold  $M$ , the map  $H_*^{\text{sm}}(M) \rightarrow H_*^{\text{dR}}(M)$  is an isomorphism.*

Let us denote the map  $H_*^{\text{sm}}(M) \rightarrow H_*^{\text{dR}}(M)$  by  $I_0$ . The proof of Theorem 3.1 is separated into two steps. Let  $d := \dim M$ .

- In Section 3.1, we define an isomorphism  $I_1 : H_{c, \text{dR}}^{d-*}(M) \cong H_*^{\text{dR}}(M)$ .
- In Section 3.2, we define an isomorphism  $I_2 : H_*^{\text{sm}}(M) \cong H_{c, \text{dR}}^{d-*}(M)$  via the Poincaré duality, and show that  $I_0 = \pm I_1 \circ I_2$ .

**3.1. Comparison with compactly supported de Rham cohomology.** Let us consider the differentiable space  $M_{\text{reg}}$  (see Example 2.2 (i)-(b)). It is easy to check that

$$(7) \quad C_*^{\text{dR}}(M_{\text{reg}}) \rightarrow \mathcal{A}_c^{d-*}(M); \quad [(U, \varphi, \omega)] \mapsto \varphi! \omega$$

is a well-defined chain map.

On the other hand, for any  $\omega \in \mathcal{A}_c^{d-*}(M)$ , let us take  $U \in \mathcal{U}$  and an orientation-preserving open embedding  $\varphi : U \rightarrow M$  such that  $\text{supp } \omega \subset \varphi(U)$ . Then,  $[(U, \varphi, \varphi^* \omega)] \in C_*^{\text{dR}}(M_{\text{reg}})$  does not depend on choices of  $U$  and  $\varphi$ . Thus, one can define a chain map

$$\mathcal{A}_c^{d-*}(M) \rightarrow C_*^{\text{dR}}(M_{\text{reg}}); \quad \omega \mapsto [(U, \varphi, \varphi^* \omega)],$$

and this is the inverse of (7). Therefore, (7) is an isomorphism of chain complexes. In particular,  $H_*^{\text{dR}}(M_{\text{reg}}) \cong H_{\text{dR},c}^{d-*}(M)$ .

$\text{id}_M : M_{\text{reg}} \rightarrow M$  is a map of differentiable spaces, as noted in Example 2.2 (i). The goal of this subsection is to prove the next proposition.

**Proposition 3.2.**  $\text{id}_M : M_{\text{reg}} \rightarrow M$  induces an isomorphism  $H_*^{\text{dR}}(M_{\text{reg}}) \cong H_*^{\text{dR}}(M)$ .

As an immediate consequence, we obtain an isomorphism  $I_1 : H_{c,\text{dR}}^{d-*}(M) \cong H_*^{\text{dR}}(M)$ .

To prove Proposition 3.2, let us take a proper  $C^\infty$ -function  $f : M \rightarrow \mathbb{R}$ , and set  $M^{(k)} := f^{-1}((-k, k))$  for any integer  $k \geq 1$ . Since  $f$  is a smooth function on  $M$  and  $M_{\text{reg}}$ , Corollary 2.11 implies isomorphisms

$$H_*^{\text{dR}}(M) \cong \varinjlim_k H_*^{\text{dR}}(M^{(k)}), \quad H_*^{\text{dR}}(M_{\text{reg}}) \cong \varinjlim_k H_*^{\text{dR}}(M_{\text{reg}}^{(k)}).$$

Here we need the following lemma.

**Lemma 3.3.** *Let  $N$  be an oriented  $C^\infty$ -manifold, and let  $W$  be an open set in  $N$  with a compact closure. Then, there exists a chain map  $J : C_*^{\text{dR}}(W) \rightarrow C_*^{\text{dR}}(N_{\text{reg}})$  such that the following diagram of chain maps commutes up to homotopy.*

$$\begin{array}{ccc} C_*^{\text{dR}}(W_{\text{reg}}) & \xrightarrow{I_{\text{reg}}} & C_*^{\text{dR}}(N_{\text{reg}}) \\ (\text{id}_W)_* \downarrow & \nearrow J & \downarrow (\text{id}_N)_* \\ C_*^{\text{dR}}(W) & \xrightarrow{I} & C_*^{\text{dR}}(N) \end{array}$$

$I_{\text{reg}}$  and  $I$  are induced by the inclusion map  $W \rightarrow N$ .

Let us apply Lemma 3.3 for  $N = M^{(k+1)}$ ,  $W = M^{(k)}$ , and take a chain map  $J^{(k)} : C_*^{\text{dR}}(M^{(k)}) \rightarrow C_*^{\text{dR}}(M_{\text{reg}}^{(k+1)})$  as in Lemma 3.3. Then,

$$\varinjlim_k H_*(J^{(k)}) : \varinjlim_k H_*^{\text{dR}}(M^{(k)}) \rightarrow \varinjlim_k H_*^{\text{dR}}(M_{\text{reg}}^{(k)})$$

is the inverse of  $H_*^{\text{dR}}(M_{\text{reg}}) \rightarrow H_*^{\text{dR}}(M)$ , thus Proposition 3.2 is proved.

To prove Lemma 3.3, we need the following lemma.

**Lemma 3.4.** *Let  $N$  be a  $C^\infty$ -manifold, and  $K$  be a compact set in  $N$ . There exists an integer  $D > 0$  and a  $C^\infty$ -map  $F : N \times \mathbb{R}^D \rightarrow N$  such that the following conditions hold.*

- For any  $z \in \mathbb{R}^D$ ,  $F_z : N \rightarrow N; x \mapsto F(x, z)$  is a diffeomorphism.
- $F_{(0, \dots, 0)} = \text{id}_N$ .
- For any  $x \in K$ ,  $\mathbb{R}^D \rightarrow N; z \mapsto F(x, z)$  is a submersion.

**Proof.** Let  $\mathcal{X}_c(N)$  denote the space of compactly supported  $C^\infty$ -vector fields on  $N$ . For any  $\xi \in \mathcal{X}_c(N)$ , let  $(\Phi_\xi^t)_{t \in \mathbb{R}}$  denote the flow generated by  $\xi$ .

Let us take a sequence  $\xi = (\xi_j)_{1 \leq j \leq D}$  on  $\mathcal{X}_c(N)$ , such that  $(\xi_j(x))_j$  spans  $T_x N$  for any  $x \in K$ . For  $z = (z_1, \dots, z_D) \in \mathbb{R}^D$ , we set  $z \cdot \xi := \sum_j z_j \xi_j$ . Let us define a

$C^\infty$ -map  $f : N \times \mathbb{R}^D \rightarrow N$  by  $f(x, z) := \Phi_{z, \xi}^1(x)$ . Then,  $\partial_z f(x, z) : \mathbb{R}^D \rightarrow T_{f(x, z)}N$  is onto for any  $x \in K$  and  $|z| < \varepsilon$  for some  $\varepsilon > 0$ . Finally, take any diffeomorphism  $g : \mathbb{R}^D \rightarrow \{z \in \mathbb{R}^D \mid |z| < \varepsilon\}$  preserving the origin. Then,  $F(x, z) := f(x, g(z))$  satisfies the requirements of the lemma.  $\square$

**Remark 3.5.** When  $N$  is a Riemannian manifold, for any  $\delta > 0$  we may further require the following condition: for any  $v \in TN$  and  $z \in \mathbb{R}^D$ ,  $|dF_z(v)| \leq (1 + \delta)|v|$ .

**Proof of Lemma 3.3.** Let us apply Lemma 3.4 for  $K = \bar{W}$ , and take an integer  $D > 0$  and  $F : N \times \mathbb{R}^D \rightarrow N$ . For any  $\varphi \in C^\infty(U, W)$ ,  $F \circ (\varphi \times \text{id}_{\mathbb{R}^D}) : U \times \mathbb{R}^D \rightarrow N$  is a submersion. We take  $\nu_D \in \mathcal{A}_c^D(\mathbb{R}^D)$  so that  $\int_{\mathbb{R}^D} \nu_D = 1$ , and define

$$J : C_*^{\text{dR}}(W) \rightarrow C_*^{\text{dR}}(N_{\text{reg}}); \quad [(U, \varphi, \omega)] \mapsto [(U \times \mathbb{R}^D, F \circ (\varphi \times \text{id}_{\mathbb{R}^D}), \omega \times \nu_D)].$$

It is easy to see that  $J$  is a well-defined chain map.

To show that  $J \circ (\text{id}_W)_*$  and  $I_{\text{reg}}$  are chain homotopic, let us take  $a, b \in C^\infty(\mathbb{R}, [0, 1])$  so that

- $a(s) = 0$  for any  $s \leq 0$ , and  $a(s) = 1$  for any  $s \geq 1$ .
- $\text{supp } b$  is compact, and there exists  $\varepsilon > 0$  such that  $b(s) = 1$  for any  $s \in [-\varepsilon, 1 + \varepsilon]$ .

For any  $(U, \varphi) \in \mathcal{P}(W_{\text{reg}})$ , we define  $\Phi : U \times \mathbb{R}^D \times \mathbb{R} \rightarrow N$  by

$$\Phi(u, z, s) := F(\varphi(u), a(s)z) = F_{a(s)z}(\varphi(u)).$$

Since  $\varphi : U \rightarrow W$  is a submersion, and  $F_{a(s)z}$  is a diffeomorphism on  $N$  for any  $(z, s) \in \mathbb{R}^D \times \mathbb{R}$ ,  $\Phi$  is also a submersion. Therefore,  $(U \times \mathbb{R}^D \times \mathbb{R}, \Phi) \in \mathcal{P}(N_{\text{reg}})$ .

Now, it is easy to see that

$$K : C_*^{\text{dR}}(W_{\text{reg}}) \rightarrow C_{*+1}^{\text{dR}}(N_{\text{reg}}); \quad [(U, \varphi, \omega)] \mapsto (-1)^{|\omega|+D}[(U \times \mathbb{R}^D \times \mathbb{R}, \Phi, \omega \times \nu_D \times b(s))]$$

is a well-defined linear map, and  $\partial K + K\partial = I_{\text{reg}} - J \circ (\text{id}_W)_*$ .

Similar arguments show that  $(\text{id}_N)_* \circ J$  and  $I$  are chain homotopic. The homotopy operator is given by exactly the same formula as  $K$ . This case is easier, since we do not have to care about the submersion condition.  $\square$

**3.2. Proof of Theorem 3.1.** Let us define an isomorphism  $I_2 : H_*^{\text{sm}}(M) \cong H_{c, \text{dR}}^{d-*}(M)$ . When  $H_{\text{dR}}^*(M)$  is finite-dimensional, it is defined as

$$I_2 : H_*^{\text{sm}}(M) \cong (H_{\text{dR}}^*(M))^* \cong H_{c, \text{dR}}^{d-*}(M).$$

The first isomorphism is defined by integrations of differential forms on smooth chains, and the second one follows from the Poincaré duality. To define  $I_2$  in the general case, let us define a set  $\mathcal{U}_M$  by

$$\mathcal{U}_M := \{\text{a relatively compact open set } U \subset M \text{ such that } \dim H_{\text{dR}}^*(U) < \infty\}.$$

Then, we define  $I_2$  by

$$I_2 : H_*^{\text{sm}}(M) \cong \varinjlim_{U \in \mathcal{U}_M} H_*^{\text{sm}}(U) \cong \varinjlim_{U \in \mathcal{U}_M} H_{c, \text{dR}}^{d-*}(U) \cong H_{c, \text{dR}}^{d-*}(M).$$

To prove that  $I_0$  is an isomorphism, it is enough to check that  $I_0 = \pm I_1 \circ I_2$ . We may assume that  $H_{\text{dR}}^*(M)$  is finite-dimensional, since the general case follows from this case by taking limits. Let us consider the following diagram:

$$\begin{array}{ccccc} H_*^{\text{sm}}(M) & \xrightarrow{\cong} & (H_{\text{dR}}^*(M))^* & \xleftarrow{\cong} & H_{c,\text{dR}}^{d-*}(M) \\ & \searrow I_0 & \uparrow & \swarrow I_1 & \\ & & H_*^{\text{dR}}(M) & & \end{array}$$

The vertical map  $H_*^{\text{dR}}(M) \rightarrow (H_{\text{dR}}^*(M))^*$  is defined by the pairing  $\langle \cdot, \cdot \rangle : H_{\text{dR}}^*(M) \otimes H_*^{\text{dR}}(M) \rightarrow \mathbb{R}$  (see Section 2.8).

To show that  $I_0 = \pm I_1 \circ I_2$ , it is enough to check that the above diagram commutes up to sign. The commutativity of the right triangle is easy to check from the definitions. The commutativity of the left triangle follows from Lemma 2.14 (the case  $m = 1$ ).

#### 4. DE RHAM CHAINS ON $C^\infty$ FREE LOOP SPACES

Let  $M$  be a closed, oriented Riemannian manifold. We abbreviate  $\mathcal{L}M := C^\infty(S^1, M)$  as  $\mathcal{L}$ . We consider the differentiable structure on  $\mathcal{L}$  as in Example 2.2 (ii). For any  $a \in (0, \infty]$ , we set  $\mathcal{L}^a := \{\gamma \in \mathcal{L} \mid \text{len}(\gamma) < a\}$ , and consider the differentiable structure as a subset of  $\mathcal{L}$  (see Example 2.2 (iii)).

Any strongly smooth map  $\sigma : \Delta^k \rightarrow \mathcal{L}^a$  is continuous with respect to the  $C^\infty$ -topology on  $\mathcal{L}^a$ . Therefore, we obtain a map  $H_*^{\text{sm}}(\mathcal{L}^a) \rightarrow H_*(\mathcal{L}^a)$ , where the RHS denotes the singular homology. On the other hand, for any differentiable space  $X$ , we defined the map  $H_*^{\text{sm}}(X) \rightarrow H_*^{\text{dR}}(X)$ . The aim of this section is to prove the following result.

**Theorem 4.1.** *For any closed, oriented Riemannian manifold  $M$  and  $a \in (0, \infty]$ , the maps  $H_*^{\text{sm}}(\mathcal{L}^a) \rightarrow H_*^{\text{dR}}(\mathcal{L}^a)$  and  $H_*^{\text{sm}}(\mathcal{L}^a) \rightarrow H_*(\mathcal{L}^a)$  are isomorphisms.*

As an immediate consequence, we obtain an isomorphism  $H_*^{\text{dR}}(\mathcal{L}^a) \cong H_*(\mathcal{L}^a)$ . The proof of Theorem 4.1 uses finite-dimensional approximations of the free loop space  $\mathcal{L}^a$ , which we explain in Section 4.1.

Recall that the rotation operator  $\Delta : H_*(\mathcal{L}^a) \rightarrow H_{*+1}(\mathcal{L}^a)$  is defined as  $\Delta(x) := -H_*(r)([S^1] \times x)$ , where  $r : S^1 \times \mathcal{L}^a \rightarrow \mathcal{L}^a$  denotes the rotation. Via isomorphisms  $H_*(S^1) \cong H_*^{\text{sm}}(S^1) \cong H_*^{\text{dR}}(S^1)$ , one can define the rotation operators on  $H_*^{\text{sm}}(\mathcal{L}^a)$  and  $H_*^{\text{dR}}(\mathcal{L}^a)$  in the same way. It is easy to see that the isomorphism  $H_*(\mathcal{L}^a) \cong H_*^{\text{sm}}(\mathcal{L}^a)$  preserves the rotation operators, since  $C_*^{\text{sm}}(X)$  is a subcomplex of  $C_*(X)$  for  $X = S^1, \mathcal{L}^a$ . The isomorphism  $H_*^{\text{sm}}(\mathcal{L}^a) \cong H_*^{\text{dR}}(\mathcal{L}^a)$  also preserves the rotation operators, since  $H_*^{\text{sm}} \rightarrow H_*^{\text{dR}}$  is functorial and commutes with the cross product (Lemma 2.13). Thus we have proved the following corollary.

**Corollary 4.2.** *The isomorphism  $H_*(\mathcal{L}^a) \cong H_*^{\text{dR}}(\mathcal{L}^a)$  preserves the rotation operators.*

**4.1. Finite-dimensional approximations of  $\mathcal{L}$ .** Let us define  $\mathcal{E} : \mathcal{L} \rightarrow \mathbb{R}$  by  $\mathcal{E}(\gamma) := \int_{S^1} |\dot{\gamma}|^2$ .  $\mathcal{E}$  is smooth as a function on the differentiable space  $\mathcal{L}$ . For any  $E \in (0, \infty]$ , we define  $\mathcal{L}^{a,E} \subset \mathcal{L}^a$  by  $\mathcal{L}^{a,E} := \{\gamma \in \mathcal{L}^a \mid \mathcal{E}(\gamma) < E\}$ .

For any positive integer  $N$ , let us define

$$\mathcal{F}_N := \{(x^j)_{0 \leq j \leq N} \in M^{N+1} \mid x^0 = x^N\}, \quad f_N : \mathcal{L} \rightarrow \mathcal{F}_N; \gamma \mapsto (\gamma(j/N))_{0 \leq j \leq N}.$$

$f_N$  is smooth as a map between differentiable spaces.

For any  $x = (x^j)_{0 \leq j \leq N} \in \mathcal{F}_N$ , let us define ( $d$  denotes the distance on  $M$ ):

$$\text{len}(x) := \sum_{1 \leq j \leq N} d(x^j, x^{j-1}), \quad \mathcal{E}(x) := N \sum_{1 \leq j \leq N} d(x^j, x^{j-1})^2.$$

For any  $a, E \in (0, \infty]$ , we define

$$\mathcal{F}_N^{a,E} := \{x \in \mathcal{F}_N \mid \text{len}(x) < a, \mathcal{E}(x) < E\}.$$

It is easy to see that  $f_N(\mathcal{L}^{a,E}) \subset \mathcal{F}_N^{a,E}$  for any  $a, E \in (0, \infty]$ .

Let  $r_M$  be the injectivity radius of  $M$  (since  $M$  is closed,  $r_M > 0$ ). For any  $p, q \in M$  such that  $d(p, q) < r_M$ , there exists a unique shortest geodesic  $\gamma : [0, 1] \rightarrow M$  such that  $\gamma(0) = p$ ,  $\gamma(1) = q$ . We denote it by  $\gamma_{pq}$ .

Suppose that  $N$  is sufficiently large, such that  $\sqrt{E/N} < r_M$ . For any  $x = (x^j)_{0 \leq j \leq N} \in \mathcal{F}_N^{a,E}$  and  $0 \leq j \leq N-1$ , there holds  $d(x^j, x^{j+1}) < \sqrt{E/N} < r_M$ . For any integer  $m \geq 1$ , we define  $y = (y^k)_{0 \leq k \leq mN} \in \mathcal{F}_{mN}^{a,E}$  by

$$y^{jm+l} := \gamma_{x^j x^{j+1}}(l/m) \quad (0 \leq j \leq N-1, 0 \leq l \leq m).$$

For any  $a' \geq a$  and  $E' \geq E$ , we define  $i_m : \mathcal{F}_N^{a,E} \rightarrow \mathcal{F}_{mN}^{a',E'}$  by  $i_m(x) := y$ . This is a  $C^\infty$ -map between  $C^\infty$ -manifolds.

**Lemma 4.3.** *For any positive real numbers  $a < a'$  and  $E < E'$ , there exists  $N(a, E, a', E')$  such that the following holds: for any integer  $N \geq N(a, E, a', E')$  and any integer  $m \geq 1$ , there exists a map  $g : \mathcal{F}_N^{a,E} \rightarrow \mathcal{F}_{mN}^{a',E'}$  such that the following diagram commutes up to homotopy ( $i$  denotes the inclusion map):*

$$\begin{array}{ccc} \mathcal{L}^{a,E} & \xrightarrow{i} & \mathcal{L}^{a',E'} \\ f_N \downarrow & \nearrow g & \downarrow f_{mN} \\ \mathcal{F}_N^{a,E} & \xrightarrow{i_m} & \mathcal{F}_{mN}^{a',E'} \end{array}$$

$g$  and the homotopies are both continuous ( $\mathcal{L}^{a,E}$ ,  $\mathcal{L}^{a',E'}$  are equipped with the  $C^\infty$ -topology) and smooth (as maps between differentiable spaces).

To prove Lemma 4.3, we need the following preliminary Lemma 4.4.

We define  $F : [0, 1] \times \{(p, q) \in M^{\times 2} \mid d(p, q) < r_M\} \rightarrow M$  by  $F(s, p, q) := \gamma_{pq}(s)$ . For any  $s \in [0, 1]$ , we define a map  $F_s$  by  $F_s(p, q) = F(s, p, q)$ . For any  $\gamma_0, \gamma_1 \in \mathcal{L}$  such that  $\max_{t \in S^1} d(\gamma_0(t), \gamma_1(t)) < r_M$ , we define  $\gamma_s \in \mathcal{L}$  by  $\gamma_s(t) := F_s(\gamma_0(t), \gamma_1(t))$ .



**Lemma 4.4.** For any  $\delta > 0$ , there exists  $r(\delta) \in (0, r_M)$  such that the following holds: if  $\gamma_0, \gamma_1 \in \mathcal{L}$  satisfy  $\max_{t \in S^1} d(\gamma_0(t), \gamma_1(t)) < r(\delta)$ , for any  $0 \leq s \leq 1$  there holds

$$\text{len}(\gamma_s) \leq (1 + \delta)((1 - s)\text{len}(\gamma_0) + s\text{len}(\gamma_1)), \quad \mathcal{E}(\gamma_s) \leq (1 + \delta)^2((1 - s)\mathcal{E}(\gamma_0) + s\mathcal{E}(\gamma_1)).$$

**Proof.** The following assertion is easy to prove by contradiction: there exists  $r(\delta) \in (0, r_M)$  such that, if  $p, q \in M$  satisfy  $d(p, q) < r(\delta)$ , then

$$|dF_s(v, w)| \leq (1 + \delta)((1 - s)|v| + s|w|) \quad (\forall v \in T_p M, \forall w \in T_q M, \forall s \in [0, 1]).$$

Take  $r(\delta) > 0$  as above. Then, if  $\gamma_0, \gamma_1 \in \mathcal{L}$  satisfy  $\max_{t \in S^1} d(\gamma_0(t), \gamma_1(t)) < r(\delta)$ , there holds  $|\dot{\gamma}_s(t)| \leq (1 + \delta)((1 - s)|\dot{\gamma}_0(t)| + s|\dot{\gamma}_1(t)|)$  for any  $s \in [0, 1]$  and  $t \in S^1$ . The lemma follows from this estimate.  $\square$

**Proof of Lemma 4.3.** Let us take  $\delta > 0$  so that  $1 + \delta < a'/a$  and  $(1 + \delta)^4 < E'/E$ . Let us also take a  $C^\infty$ -function  $\mu : [0, 1] \rightarrow [0, 1]$  with the following properties:

- $0 \leq \mu'(t) \leq 1 + \delta$  for any  $t \in [0, 1]$ .
- $\mu(j/m) = j/m$  for any integer  $0 \leq j \leq m$ .
- $\mu$  is constant on some neighborhoods of 0 and 1.

Let us take an integer  $N$  so that  $\sqrt{E/N} < r_M$ . For any  $x = (x^j)_{0 \leq j \leq N} \in \mathcal{F}_N^{a,E}$ , we define  $\gamma \in \mathcal{L}$  by

$$\gamma((j + t)/N) := \gamma_{x^j, x^{j+1}}(\mu(t)) \quad (0 \leq j \leq N - 1, 0 \leq t \leq 1).$$

Then  $\gamma$  satisfies  $\text{len}(\gamma) = \text{len}(x) < a'$  and  $\mathcal{E}(\gamma) \leq (1 + \delta)^2 \mathcal{E}(x) < E'$ , thus one can define  $g : \mathcal{F}_N^{a,E} \rightarrow \mathcal{L}^{a',E'}$  by  $g(x) := \gamma$ . It is clear that  $f_{mN} \circ g = i_m$ .  $g$  is both smooth (as a map between differentiable spaces) and continuous ( $\mathcal{L}^{a',E'}$  is equipped with the  $C^\infty$ -topology).

Let us define a homotopy between  $i$  and  $g \circ f_N$ . For any  $\gamma \in \mathcal{L}^{a,E}$ , we set  $\gamma_0 := \gamma$ , and  $\gamma_1 := g \circ f_N(\gamma)$ . Then  $\gamma_1$  satisfies  $\text{len}(\gamma_1) < a$  and  $\mathcal{E}(\gamma_1) < (1 + \delta)^2 E$ .

When  $N$  is sufficiently large,  $\max_{t \in S^1} d(\gamma_0(t), \gamma_1(t)) \leq r(\delta)$ . Thus, for any  $0 \leq s \leq 1$ ,  $\gamma_s$  satisfies  $\text{len}(\gamma_s) < (1 + \delta)a < a'$  and  $\mathcal{E}(\gamma_s) < (1 + \delta)^4 E < E'$ .

Finally, take  $\nu \in C^\infty(\mathbb{R}, [0, 1])$  so that  $\nu(s) = 0$  for any  $s \leq 0$ , and  $\nu(s) = 1$  for any  $s \geq 1$ . Then,  $h : \mathcal{L}^{a,E} \times \mathbb{R} \rightarrow \mathcal{L}^{a',E'}; (\gamma, s) \mapsto \gamma_{\nu(s)}$  is a homotopy between  $i$  and  $g \circ f_N$ .  $h$  is both smooth and continuous.  $\square$

**Remark 4.5.** As is clear from the above proof, Lemma 4.3 holds even when  $a = a' = \infty$ . In this case, we denote  $N(a, E, a', E')$  by  $N(E, E')$ .

**4.2. Proof of Theorem 4.1.** Let us take strictly increasing sequences of positive real numbers  $(a_j)_{j \geq 1}$  and  $(E_j)_{j \geq 1}$ , such that  $\lim_{j \rightarrow \infty} a_j = a$ ,  $\lim_{j \rightarrow \infty} E_j = \infty$ . Then,  $(\mathcal{L}^{a_j, E_j})_{j \geq 1}$  is an increasing sequence of open sets (with respect to the  $C^\infty$ -topology) of  $\mathcal{L}^a$ , and

$\bigcup_{j \geq 1} \mathcal{L}^{a_j, E_j} = \mathcal{L}^a$ . Thus, we have isomorphisms

$$\varinjlim_j H_*(\mathcal{L}^{a_j, E_j}) \cong H_*(\mathcal{L}^a), \quad \varinjlim_j H_*^{\text{sm}}(\mathcal{L}^{a_j, E_j}) \cong H_*^{\text{sm}}(\mathcal{L}^a).$$

Since the length functional on  $\mathcal{L}$  is approximately smooth, Corollary 2.11 implies  $\varinjlim_j H_*^{\text{dR}}(\mathcal{L}^{a_j, E_j}) \cong H_*^{\text{dR}}(\mathcal{L}^a)$ . Thus, it is enough to show that the following maps are isomorphisms:

$$\varinjlim_j H_*^{\text{sm}}(\mathcal{L}^{a_j, E_j}) \rightarrow \varinjlim_j H_*(\mathcal{L}^{a_j, E_j}), \quad \varinjlim_j H_*^{\text{sm}}(\mathcal{L}^{a_j, E_j}) \rightarrow \varinjlim_j H_*^{\text{dR}}(\mathcal{L}^{a_j, E_j}).$$

Now we apply Lemma 4.3 for each  $j$ . Let us take a sequence  $(N_j)_{j \geq 1}$  of positive integers so that  $N_j \geq N(a_j, E_j, a_{j+1}, E_{j+1})$  and  $N_j | N_{j+1}$  for every  $j$ . Then, there exists a map  $g_j : \mathcal{F}_{N_j}^{a_j, E_j} \rightarrow \mathcal{L}^{a_{j+1}, E_{j+1}}$  such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} \mathcal{L}^{a_j, E_j} & \xrightarrow{\quad} & \mathcal{L}^{a_{j+1}, E_{j+1}} \\ f_{N_j} \downarrow & \nearrow g_j & \downarrow f_{N_{j+1}} \\ \mathcal{F}_{N_j}^{a_j, E_j} & \xrightarrow{\quad} & \mathcal{F}_{N_{j+1}}^{a_{j+1}, E_{j+1}} \end{array}$$

Then  $\varinjlim_j H_*(f_{N_j}) : \varinjlim_j H_*(\mathcal{L}^{a_j, E_j}) \rightarrow \varinjlim_j H_*(\mathcal{F}_{N_j}^{a_j, E_j})$  is an isomorphism, since  $\varinjlim_j H_*(g_j)$  is its inverse. The same argument also works for  $H_*^{\text{sm}}$  and  $H_*^{\text{dR}}$ , and we obtain isomorphisms

$$\varinjlim_j H_*^{\text{sm}}(\mathcal{L}^{a_j, E_j}) \cong \varinjlim_j H_*^{\text{sm}}(\mathcal{F}_{N_j}^{a_j, E_j}), \quad \varinjlim_j H_*^{\text{dR}}(\mathcal{L}^{a_j, E_j}) \cong \varinjlim_j H_*^{\text{dR}}(\mathcal{F}_{N_j}^{a_j, E_j}).$$

These isomorphisms fit into the following commutative diagram:

$$\begin{array}{ccccc} \varinjlim_j H_*(\mathcal{L}^{a_j, E_j}) & \longleftarrow & \varinjlim_j H_*^{\text{sm}}(\mathcal{L}^{a_j, E_j}) & \longrightarrow & \varinjlim_j H_*^{\text{dR}}(\mathcal{L}^{a_j, E_j}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \varinjlim_j H_*(\mathcal{F}_{N_j}^{a_j, E_j}) & \longleftarrow & \varinjlim_j H_*^{\text{sm}}(\mathcal{F}_{N_j}^{a_j, E_j}) & \longrightarrow & \varinjlim_j H_*^{\text{dR}}(\mathcal{F}_{N_j}^{a_j, E_j}). \end{array}$$

Since  $\mathcal{F}_{N_j}^{a_j, E_j}$  is an oriented *finite-dimensional* manifold for every  $j$ ,  $H_*^{\text{sm}}(\mathcal{F}_{N_j}^{a_j, E_j}) \rightarrow H_*(\mathcal{F}_{N_j}^{a_j, E_j})$  and  $H_*^{\text{sm}}(\mathcal{F}_{N_j}^{a_j, E_j}) \rightarrow H_*^{\text{dR}}(\mathcal{F}_{N_j}^{a_j, E_j})$  are isomorphisms. Thus, the horizontal maps on the second row are isomorphisms. Therefore, the horizontal maps on the first row are also isomorphisms. This completes the proof of Theorem 4.1.

## 5. MOORE LOOPS WITH MARKED POINTS

Constructions of string topology operations (e.g. the loop product) require (at least) two steps:

- Fiber products of (de Rham) chains of the loop space with respect to evaluation maps.

- Concatenations of loops.

The differentiable space  $\mathcal{L} = \mathcal{L}M$ , which we studied in the previous section, is not adequate for both steps. To avoid this trouble, in this section we introduce *Moore loops with marked points*.

We explain the plan of this section. Let  $M$  denote a closed, oriented Riemannian manifold of dimension  $d$ . In Section 5.1, we introduce the space  $\Pi$ , which consists of Moore paths on  $M$ . In Section 5.2, we introduce the space  $\mathcal{L}_k$ , which consists of Moore loops with  $k + 1$  marked points. We define two differentiable structures on  $\mathcal{L}_k$ , and denote the resulting differentiable spaces by  $\mathcal{L}_k$  and  $\mathcal{L}_{k,\text{reg}}$ . The latter is adequate to define fiber products on de Rham chain complexes, and we show that a sequence of de Rham chain complexes  $(C_{*+d}^{\text{dR}}(\mathcal{L}_{k,\text{reg}}))_{k \geq 0}$  has a natural structure of a cyclic dg operad with a multiplication and a unit (see Definitions 5.6, 5.7). This is a key geometric input in the definition of the chain complex  $C_*^{\mathcal{L}M}$ , which we explain in Section 6. Sections 5.3–5.5 are devoted to proofs of technical Lemmas 5.8, 5.9, 5.10, which we state in the end of Section 5.2.

**5.1. Moore paths.** Let  $M$  be a closed, oriented  $C^\infty$ -manifold. We define the set of Moore paths on  $M$  as follows:

$$\Pi := \{(\gamma, T) \mid T \in \mathbb{R}_{\geq 0}, \gamma \in C^\infty([0, T], M), \gamma^{(m)}(0) = \gamma^{(m)}(T) = 0 \ (\forall m \geq 1)\}.$$

$\gamma^{(m)}$  denotes the  $m$ -th derivative of  $\gamma$ . The last condition is required to take concatenations of  $C^\infty$ -paths. We define  $e_0, e_1 : \Pi \rightarrow M$  by  $e_0(\gamma, T) := \gamma(0)$ ,  $e_1(\gamma, T) := \gamma(T)$ . For any  $p \in M$ , let us define a map  $\gamma_p$  and  $c_p \in \Pi$  by

$$\gamma_p : \{0\} \rightarrow M; \quad 0 \mapsto p, \quad c_p := (\gamma_p, 0) \in \Pi.$$

The concatenation map  $\Pi_{e_0 \times e_1} \Pi \rightarrow \Pi; (\gamma_0, T_0, \gamma_1, T_1) \mapsto (\gamma_0 * \gamma_1, T_0 + T_1)$  is defined by

$$\gamma_0 * \gamma_1(t) := \begin{cases} \gamma_0(t) & (0 \leq t \leq T_0) \\ \gamma_1(t - T_0) & (T_0 \leq t \leq T_0 + T_1). \end{cases}$$

Functionals  $\text{len}$  and  $\mathcal{E}$  on  $\Pi$  are defined by

$$\text{len}(\gamma, T) := \int_0^T |\dot{\gamma}|, \quad \mathcal{E}(\gamma, T) := \int_0^T |\dot{\gamma}|^2.$$

To define a differentiable structure on  $\Pi$ , we need the following definition.

**Definition 5.1.** Let  $X$  and  $Y$  be  $C^\infty$ -manifolds, and  $S$  be any subset of  $X$ . Then, a map  $f : S \rightarrow Y$  is of class  $C^\infty$ , if there exists an open neighborhood  $U$  of  $S \subset X$  and a  $C^\infty$ -map  $\tilde{f} : U \rightarrow Y$  such that  $\tilde{f}|_S = f$ .

We define two differentiable structures on  $\Pi$ , and denote the resulting differentiable spaces as  $\Pi$  and  $\Pi_{\text{reg}}$ . The set of plots  $\mathcal{P}(\Pi)$  and  $\mathcal{P}(\Pi_{\text{reg}})$  are defined as follows:

- Let  $U \in \mathcal{U}$  and  $\varphi : U \rightarrow \Pi$ . Denote  $\varphi(u) = (\gamma(u), T(u))$  for any  $u \in U$ . Then,  $(U, \varphi) \in \mathcal{P}(\Pi)$  if  $T \in C^\infty(U)$  and

$$\tilde{U} := \{(u, t) \mid u \in U, 0 \leq t \leq T(u)\} \rightarrow M; \quad (u, t) \mapsto \gamma(u)(t)$$

is of class  $C^\infty$  in the sense of Definition 5.1 ( $X := U \times \mathbb{R}$ ,  $S := \tilde{U}$ ).

- $\mathcal{P}(\Pi_{\text{reg}})$  consists of  $(U, \varphi) \in \mathcal{P}(\Pi)$  such that  $e_j \circ \varphi : U \rightarrow M$  is a submersion for  $j = 0, 1$ .

The identity map  $\text{id}_\Pi : \Pi_{\text{reg}} \rightarrow \Pi$  is smooth as a map of differentiable spaces. The functional  $\mathcal{E}$  is smooth, and  $\text{len}$  is approximately smooth with both differentiable structures  $(\Pi \text{ and } \Pi_{\text{reg}})$ . The goal of this subsection is to prove the next lemma.

**Lemma 5.2.** *The concatenation maps*

$$\Pi_{e_0 \times e_1} \Pi \rightarrow \Pi, \quad \Pi_{\text{reg } e_0 \times e_1} \Pi_{\text{reg}} \rightarrow \Pi_{\text{reg}}$$

*are smooth as maps of differentiable spaces.*

**Remark 5.3.** Set theoretically, the two maps in Lemma 5.2 are same. However, differentiable structures on the domain and the target of the map are different.

First we prove the following lemma, which is a subtle point of the proof.

**Lemma 5.4.** *Let  $U$  be a  $C^\infty$ -manifold, and  $T \in C^\infty(U, \mathbb{R}_{\geq 0})$ . Let  $\tilde{U} := \{(u, t) \mid u \in U, 0 \leq t \leq T(u)\} \subset U \times \mathbb{R}$ , and suppose that  $f : U \rightarrow \mathbb{R}$  satisfies the following conditions:*

- *$f$  is of class  $C^\infty$  in the sense of Definition 5.1.*
- *For any  $u \in U$  and an integer  $m \geq 0$ ,  $\partial_t^m f(u, 0) = \partial_t^m f(u, T(u)) = 0$ .*

*Then,  $\tilde{f} : U \times \mathbb{R} \rightarrow \mathbb{R}$  defined by  $\tilde{f}(u, t) = \begin{cases} f(u, t) & ((u, t) \in \tilde{U}) \\ 0 & ((u, t) \notin \tilde{U}) \end{cases}$  is of class  $C^\infty$ .*

**Proof.** Let  $U_0 := \{u \in U \mid T \equiv 0 \text{ on a neighborhood of } u\}$ . A nontrivial part of the lemma is to check differentiability of  $\tilde{f}$  at  $(u, 0)$  such that  $T(u) = 0$  and  $u \notin U_0$ .

Since  $f$  is of class  $C^\infty$ , there exists a  $C^\infty$ -function  $F$ , which is defined on an open neighborhood of  $\tilde{U}$  and  $F|_{\tilde{U}} = f$ . Let us prove  $\partial^\alpha F(u, 0) = 0$  for any  $\alpha$  and  $u \notin U_0$ . It is obvious when  $T(u) > 0$ , since  $\partial_t^m F(v, 0) = \partial_t^m f(v, 0) = 0$  for any  $m \geq 0$  and  $v$  which is near  $u$ . Even when  $T(u) = 0$ , there exists a sequence  $(u_j)_{j \geq 1}$  such that  $\lim_{j \rightarrow \infty} u_j = u$  and  $T(u_j) > 0$ , since  $u \notin U_0$ . Thus,  $\partial^\alpha F(u, 0) = \lim_{j \rightarrow \infty} \partial^\alpha F(u_j, 0) = 0$ .

To prove the lemma, it is enough to prove the claim  $\text{Cl}(m)$  for every integer  $m \geq 0$ :

$\text{Cl}(m)$ : For any  $\alpha$  such that  $|\alpha| = m$ ,  $\partial^\alpha \tilde{f}$  is totally differentiable, and  $D(\partial^\alpha \tilde{f})(u, t) = 0$  unless  $0 < t < T(u)$ .

Let us prove  $\text{Cl}(0)$ . It is enough to check  $D\tilde{f}(u, 0) = 0$  for any  $u$  such that  $T(u) = 0$  and  $u \notin U_0$ . If this does not hold, there exists a sequence  $(u_j, t_j)_{j \geq 1}$  such that  $\lim_{j \rightarrow \infty} (u_j, t_j) = (u, 0)$  and  $\liminf_j |\tilde{f}(u_j, t_j) - \tilde{f}(u, 0)| / |(u_j, t_j) - (u, 0)| > 0$ . Since  $\tilde{f}(u, 0) = 0$ , we may assume  $\tilde{f}(u_j, t_j) \neq 0$ . Then,  $0 < t_j < T(u_j)$ , and thus,  $F(u_j, t_j) = \tilde{f}(u_j, t_j)$ . On the other hand  $F(u, 0) = 0$ , and thus,  $\liminf_j |F(u_j, t_j) - F(u, 0)| / |(u_j, t_j) - (u, 0)| > 0$ . This contradicts  $DF(u, 0) = 0$ , and  $\text{Cl}(0)$  is proved.

Let us prove  $\text{Cl}(m-1) \implies \text{Cl}(m)$ . It is enough to check  $D(\partial^\alpha \tilde{f})(u, 0) = 0$  for any  $u$  such that  $T(u) = 0$  and  $u \notin U_0$ . If this does not hold, there exists a sequence  $(u_j, t_j)_{j \geq 1}$  such that  $\lim_{j \rightarrow \infty} (u_j, t_j) = (u, 0)$  and  $\liminf_j |\partial^\alpha \tilde{f}(u_j, t_j) - \partial^\alpha \tilde{f}(u, 0)| / |(u_j, t_j) - (u, 0)| > 0$ .

By  $\text{Cl}(m-1)$ ,  $\partial^\alpha \tilde{f}(u, 0) = 0$ . Thus, we may assume that  $\partial^\alpha \tilde{f}(u_j, t_j) \neq 0$ . By  $\text{Cl}(m-1)$ , this implies  $0 < t_j < T(u_j)$ , and thus,  $\partial^\alpha F(u_j, t_j) = \partial^\alpha \tilde{f}(u_j, t_j)$ . On the other hand  $\partial^\alpha F(u, 0) = 0$ , and thus,  $\liminf_j |\partial^\alpha F(u_j, t_j) - \partial^\alpha F(u, 0)| / |(u_j, t_j) - (u, 0)| > 0$ . This contradicts  $D(\partial^\alpha F)(u, 0) = 0$ , and  $\text{Cl}(m)$  is proved.  $\square$

**Proof of Lemma 5.2.** We first prove that  $\Pi_{e_0 \times_{e_1} \Pi} \rightarrow \Pi$  is smooth. Let  $(U, \varphi) \in \mathcal{P}(\Pi_{e_0 \times_{e_1} \Pi})$ , and denote  $\varphi(u) = (\gamma_0(u), T_0(u), \gamma_1(u), T_1(u))$ . We need to show that  $U \rightarrow \Pi; u \mapsto (\gamma_0(u) * \gamma_1(u), T_0(u) + T_1(u))$  is a plot on  $\Pi$ . It is enough to show that

$$\Gamma_{01} : \{(u, t) \mid u \in U, 0 \leq t \leq T_0(u) + T_1(u)\} \rightarrow M; \quad (u, t) \mapsto \gamma_0(u) * \gamma_1(u)(t)$$

is of class  $C^\infty$  in the sense of Definition 5.1. We may assume that  $M$  is embedded in  $\mathbb{R}^N$  for some integer  $N$ . For  $j = 0, 1$ , we define  $\delta_j : U \times \mathbb{R} \rightarrow \mathbb{R}^N$  by

$$\delta_j(u, t) := \begin{cases} \partial_t \gamma_j(u, t) & (0 \leq t \leq T_j(u)) \\ 0 & (\text{otherwise}). \end{cases}$$

Since  $\partial_t^m \gamma_j(u, 0) = \partial_t^m \gamma_j(u, T_j(u)) = 0$  for any  $u \in U$  and  $m \geq 1$ , Lemma 5.4 shows that  $\delta_j \in C^\infty(U \times \mathbb{R}, \mathbb{R}^N)$ . Let us define  $\delta_{01}, \widetilde{\Gamma}_{01} \in C^\infty(U \times \mathbb{R}, \mathbb{R}^N)$  by

$$\delta_{01}(u, t) := \delta_0(u, t) + \delta_1(u, t - T_0(u)), \quad \widetilde{\Gamma}_{01}(u, t) := \gamma_0(u)(0) + t \int_0^1 \delta_{01}(u, ts) ds.$$

Then, it is easy to see that  $\widetilde{\Gamma}_{01}(u, t) = \Gamma_{01}(u, t)$  for any  $u \in U$  and  $0 \leq t \leq T_0(u) + T_1(u)$ . This shows that  $\Gamma_{01}$  is of class  $C^\infty$ , hence  $\Pi_{e_0 \times_{e_1} \Pi} \rightarrow \Pi$  is smooth.

Now, it is easy to see that  $\Pi_{\text{reg } e_0 \times_{e_1} \Pi_{\text{reg}}} \rightarrow \Pi_{\text{reg}}$  is smooth, since any  $(\Gamma_0, \Gamma_1) \in \Pi_{e_0 \times_{e_1} \Pi}$  satisfies  $e_0(\Gamma_0 * \Gamma_1) = e_0(\Gamma_0)$  and  $e_1(\Gamma_0 * \Gamma_1) = e_1(\Gamma_1)$ .  $\square$

**5.2. Moore loops with marked points.** For any integer  $k \geq 0$ , we define

$$\bar{\mathcal{L}}_k := \{(\Gamma_0, \dots, \Gamma_k) \in \Pi^{k+1} \mid e_1(\Gamma_j) = e_0(\Gamma_{j+1}) (0 \leq j \leq k-1), e_1(\Gamma_k) = e_0(\Gamma_0)\}.$$

In particular,  $\bar{\mathcal{L}}_0 = \{\Gamma \in \Pi \mid e_1(\Gamma) = e_0(\Gamma)\}$ . For any  $0 \leq i \leq k$ , we define

$$e_i : \bar{\mathcal{L}}_k \rightarrow M; \quad (\Gamma_0, \dots, \Gamma_k) \mapsto e_0(\Gamma_i).$$

We also define  $i_k : M \rightarrow \bar{\mathcal{L}}_k$  by  $i_k(p) := (c_p, \dots, c_p) (\forall p \in M)$ .

**Remark 5.5.** We can identify  $\bar{\mathcal{L}}_k$  with

$$\{(\gamma, t_1, \dots, t_k, T) \mid T \in \mathbb{R}_{\geq 0}, \gamma \in C^\infty([0, T], M), 0 \leq t_1 \leq \dots \leq t_k \leq T, \\ \gamma(0) = \gamma(T), \quad \gamma^{(m)}(0) = \gamma^{(m)}(T) = \gamma^{(m)}(t_j) = 0 (\forall m \geq 1, 1 \leq j \leq k)\}.$$

Indeed, for any  $(\gamma_j, T_j)_{0 \leq j \leq k} \in \bar{\mathcal{L}}_k$ , one can assign  $(\gamma_0 * \dots * \gamma_k, T_1, T_1 + T_2, \dots, T_1 + \dots + T_k)$ .

Recall that we defined two differentiable structures on  $\Pi$ , and denote the resulting differentiable spaces  $\Pi$  and  $\Pi_{\text{reg}}$ . Since  $\bar{\mathcal{L}}_k$  is a subset of  $\Pi^{k+1}$ , we can define two differentiable structures on  $\bar{\mathcal{L}}_k$  (see Example 2.2 (iii) and (iv)). We denote the resulting differentiable spaces by  $\bar{\mathcal{L}}_k$  and  $\bar{\mathcal{L}}_{k, \text{reg}}$ .

Let us define  $\text{len} : \bar{\mathcal{L}}_k \rightarrow \mathbb{R}$  by  $\text{len}(\Gamma_1, \dots, \Gamma_k) := \text{len}(\Gamma_1) + \dots + \text{len}(\Gamma_k)$ . The function  $\text{len}$  is approximately smooth with both differentiable structures  $(\bar{\mathcal{L}}_k$  and  $\bar{\mathcal{L}}_{k,\text{reg}}$ ).

For any  $a \in (0, \infty]$ , we define  $\bar{\mathcal{L}}_k^a := \{(\Gamma_1, \dots, \Gamma_k) \in \bar{\mathcal{L}}_k \mid \text{len}(\Gamma_1, \dots, \Gamma_k) < a\}$ . We define differentiable structures on  $\bar{\mathcal{L}}_k^a$  as a subspace of  $\bar{\mathcal{L}}_k$  and  $\bar{\mathcal{L}}_{k,\text{reg}}$ , and denote the resulting differentiable spaces as  $\bar{\mathcal{L}}_k^a$  and  $\bar{\mathcal{L}}_{k,\text{reg}}^a$ . The chain maps  $C_*^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}^a) \rightarrow C_*^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}})$  and  $C_*^{\text{dR}}(\bar{\mathcal{L}}_k^a) \rightarrow C_*^{\text{dR}}(\bar{\mathcal{L}}_k)$  are injective.

For any  $1 \leq i \leq k$  and  $l \geq 0$ , we define a map

$$c_{k,i,l} : \bar{\mathcal{L}}_{k,\text{reg}} \times_{e_i \times e_0} \bar{\mathcal{L}}_{l,\text{reg}} \rightarrow \bar{\mathcal{L}}_{k+l-1,\text{reg}}$$

by

$$c_{k,i,l}(\Gamma_0, \dots, \Gamma_k, \Gamma'_0, \dots, \Gamma'_l) := \begin{cases} (\Gamma_0, \dots, \Gamma_{i-2}, \Gamma_{i-1} * \Gamma'_0, \Gamma'_1, \dots, \Gamma'_{l-1}, \Gamma'_l * \Gamma_i, \Gamma_{i+1}, \dots, \Gamma_k) & (l \geq 1), \\ (\Gamma_0, \dots, \Gamma_{i-2}, \Gamma_{i-1} * \Gamma'_0 * \Gamma_i, \Gamma_{i+1}, \dots, \Gamma_k) & (l = 0). \end{cases}$$

Then,  $c_{k,i,l}$  is a smooth map by Lemma 5.2. Also,

$$R_k : \bar{\mathcal{L}}_{k,\text{reg}} \rightarrow \bar{\mathcal{L}}_{k,\text{reg}}; \quad (\Gamma_0, \dots, \Gamma_k) \mapsto (\Gamma_1, \dots, \Gamma_k, \Gamma_0)$$

is a smooth map.

Now we need some definitions on algebraic operads. Our notion of a multiplication of an operad (Definition 5.7) is similar to the one in [16].

**Definition 5.6.** Let  $\mathcal{O} = (\mathcal{O}(k)_*)_{k \geq 0}$  be a nonsymmetric dg operad. A *cyclic structure* on  $\mathcal{O}$  is a sequence  $(\tau_k)_{k \geq 0}$  with the following properties.

- For any  $k \geq 0$ ,  $\tau_k$  is a chain map on  $\mathcal{O}(k)_*$  of degree 0, satisfying  $\tau_k^{k+1} = \text{id}_{\mathcal{O}(k)_*}$ .
- $1_{\mathcal{O}} \in \mathcal{O}(1)_0$  is cyclically invariant, i.e.  $\tau_1(1_{\mathcal{O}}) = 1_{\mathcal{O}}$ .
- For any  $1 \leq i \leq k$ ,  $l \geq 0$ ,  $x \in \mathcal{O}(k)_*$  and  $y \in \mathcal{O}(l)_*$ , there holds

$$\tau_{k+l-1}(x \circ_i y) = \begin{cases} \tau_k x \circ_{i-1} y & (i \geq 2) \\ (-1)^{|x||y|} \tau_l y \circ_l \tau_k x & (i = 1, l \geq 1) \\ \tau_k^2 x \circ_k y & (i = 1, l = 0). \end{cases}$$

A pair  $(\mathcal{O}, (\tau_k)_{k \geq 0})$  is called a *nonsymmetric cyclic dg operad*.

**Definition 5.7.** Let  $\mathcal{O} = (\mathcal{O}(k)_*)_{k \geq 0}$  be a nonsymmetric dg operad.  $\mu \in \mathcal{O}(2)_0$  is called a *multiplication* of  $\mathcal{O}$ , if  $\partial\mu = 0$  and  $\mu \circ_1 \mu = \mu \circ_2 \mu$ .  $\varepsilon \in \mathcal{O}(0)_0$  is called a *unit* of  $\mu$ , if  $\partial\varepsilon = 0$  and  $\mu \circ_1 \varepsilon = \mu \circ_2 \varepsilon = 1_{\mathcal{O}}$ .

Let  $(\mathcal{O}, \mu, \varepsilon)$  be a nonsymmetric dg operad with a multiplication and a unit. Then,  $(\mathcal{O}(k)_*)_{k \geq 0}$  has the structure of a cosimplicial chain complex with chain maps

$$\delta_{k,i} : \mathcal{O}(k-1)_* \rightarrow \mathcal{O}(k)_* \quad (0 \leq i \leq k), \quad \sigma_{k,i} : \mathcal{O}(k+1)_* \rightarrow \mathcal{O}(k)_* \quad (0 \leq i \leq k)$$

defined as

$$\delta_{k,i}(x) := \begin{cases} \mu \circ_2 x & (i = 0) \\ x \circ_i \mu & (1 \leq i \leq k-1) \\ \mu \circ_1 x & (i = k). \end{cases} \quad \sigma_{k,i}(x) := x \circ_{i+1} \varepsilon.$$

Also, suppose that  $\mathcal{O}$  has a cyclic structure  $(\tau_k)_{k \geq 0}$ , and  $\mu \in \mathcal{O}(2)_0$  is cyclically invariant, i.e.  $\tau_2(\mu) = \mu$ . Then,  $\mathcal{O}$  is a cocyclic chain complex.

Now let us return to the Moore loop spaces. For any  $k \geq 0$ , let  $\mathcal{CL}(k)_* := C_{*+d}^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}})$ . Then,  $\mathcal{CL} := (\mathcal{CL}(k)_*)_{k \geq 0}$  has the structure of a nonsymmetric cyclic dg operad with a multiplication and a unit, defined as follows.

- For any  $1 \leq i \leq k$  and  $l \geq 0$ , we define  $\circ_i : \mathcal{CL}(k)_* \otimes \mathcal{CL}(l)_* \rightarrow \mathcal{CL}(k+l-1)_*$  by  $u \circ_i v := (c_{k,i,l})_*(u_{e_i} \times_{e_0} v)$ .
- For any  $k \geq 0$ , we define  $\tau_k : \mathcal{CL}(k)_* \rightarrow \mathcal{CL}(k)_*$  by  $\tau_k := (R_k)_*$ . We define  $\tau_0$  to be the identity map on  $\mathcal{CL}(0)_*$ .
- Let us take  $M' \in \mathcal{U}$  and an orientation-preserving diffeomorphism  $\varphi : M' \rightarrow M$ . Then, we define

$$\begin{aligned} \varepsilon &:= [(M', i_0 \circ \varphi, 1)] \in \mathcal{CL}(0)_0, & 1_{\mathcal{CL}} &:= [(M', i_1 \circ \varphi, 1)] \in \mathcal{CL}(1)_0, \\ \mu &:= [(M', i_2 \circ \varphi, 1)] \in \mathcal{CL}(2)_0. \end{aligned}$$

It is easy to check that the elements  $\varepsilon$ ,  $1_{\mathcal{CL}}$  and  $\mu$  are well-defined, i.e. not depend on choices of  $M'$  and  $\varphi$ . Also,  $1_{\mathcal{CL}}$  and  $\mu$  are cyclically invariant.

In particular,  $(C_{*+d}^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}))_{k \geq 0}$  has the structure of a cocyclic chain complex. Notice that this structure preserves the length filtration. Namely, for any  $a \in (0, \infty]$ ,  $(C_{*+d}^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}^a))_{k \geq 0}$  has the structure of a cocyclic chain complex.

The rest of this section is devoted to proofs of the following lemmas.

**Lemma 5.8.** *For any  $k \geq 0$ , the identity map  $\bar{\mathcal{L}}_{k,\text{reg}}^a \rightarrow \bar{\mathcal{L}}_k^a$  induces an isomorphism  $H_*^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}^a) \cong H_*^{\text{dR}}(\bar{\mathcal{L}}_k^a)$ .*

Let us recall notations  $\mathcal{L} = \mathcal{L}M := C^\infty(S^1, M)$  and  $\mathcal{L}^a := \{\gamma \in \mathcal{L} \mid \text{len}(\gamma) < a\}$ .

**Lemma 5.9.** *For any  $k \geq 0$ , let us define*

$$\mathcal{L}_k^a := \{(\gamma, t_1, \dots, t_k) \in \mathcal{L}^a \times \Delta^k \mid \gamma^{(m)}(0) = \gamma^{(m)}(t_j) = 0 \quad (1 \leq \forall j \leq k, \forall m \geq 1)\},$$

*and consider the differentiable structure on  $\mathcal{L}_k^a$  as a subset of  $\mathcal{L}^a \times \Delta^k$ . Then, the inclusion map  $\mathcal{L}_k^a \rightarrow \mathcal{L}^a \times \Delta^k$  induces an isomorphism  $H_*^{\text{dR}}(\mathcal{L}_k^a) \cong H_*^{\text{dR}}(\mathcal{L}^a \times \Delta^k)$ .*

**Lemma 5.10.** *For any  $k \geq 0$ , let us define the map*

$$\mathcal{L}_k^a \rightarrow \bar{\mathcal{L}}_k^a; \quad (\gamma, t_1, \dots, t_k) \mapsto (\gamma_j, T_j)_{0 \leq j \leq k}$$

*by  $T_j := t_{j+1} - t_j$ ,  $\gamma_j(t) := \gamma(t - t_j)$  (we set  $t_0 = 0$ ,  $t_{k+1} = 1$ ). Then, the map  $\mathcal{L}_k^a \rightarrow \bar{\mathcal{L}}_k^a$  is smooth, and induces an isomorphism  $H_*^{\text{dR}}(\mathcal{L}_k^a) \cong H_*^{\text{dR}}(\bar{\mathcal{L}}_k^a)$ .*

Summarizing these lemmas, we have the following zig-zag of the quasi-isomorphisms:

$$(8) \quad C_*^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}^a) \longrightarrow C_*^{\text{dR}}(\bar{\mathcal{L}}_k^a) \longleftarrow C_*^{\text{dR}}(\mathcal{L}_k^a) \longrightarrow C_*^{\text{dR}}(\mathcal{L}^a \times \Delta^k).$$

The sequences  $(C_*^{\text{dR}}(\bar{\mathcal{L}}_k^a))_{k \geq 0}$ ,  $(C_*^{\text{dR}}(\mathcal{L}_k^a))_{k \geq 0}$ ,  $(C_*^{\text{dR}}(\mathcal{L}^a \times \Delta^k))_{k \geq 0}$  have natural structures of cocyclic chain complexes, and (8) induces quasi-isomorphisms of these cocyclic chain complexes.

**5.3. Proof of Lemma 5.8.** Let us take an increasing sequence of positive real numbers  $(a_j)_{j \geq 1}$ , such that  $\lim_{j \rightarrow \infty} a_j = a$ . Since the length functional is approximately smooth on  $\bar{\mathcal{L}}_k$  and  $\bar{\mathcal{L}}_{k,\text{reg}}$ , Corollary 2.11 implies

$$\lim_j H_*^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}^{a_j}) \cong H_*^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}^a), \quad \lim_j H_*^{\text{dR}}(\bar{\mathcal{L}}_k^{a_j}) \cong H_*^{\text{dR}}(\bar{\mathcal{L}}_k^a).$$

Now, the key technical step is the next lemma.

**Lemma 5.11.** *For any integer  $j \geq 1$ , there exists a chain map  $J : C_*^{\text{dR}}(\bar{\mathcal{L}}_k^{a_j}) \rightarrow C_*^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}^{a_{j+1}})$  such that the following diagram commutes up to chain homotopy:*

$$\begin{array}{ccc} C_*^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}^{a_j}) & \xrightarrow{(\text{id}_j)_*} & C_*^{\text{dR}}(\bar{\mathcal{L}}_k^{a_j}) \\ (I_{\text{reg}})_* \downarrow & \swarrow J & \downarrow I_* \\ C_*^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}^{a_{j+1}}) & \xrightarrow{(\text{id}_{j+1})_*} & C_*^{\text{dR}}(\bar{\mathcal{L}}_k^{a_{j+1}}). \end{array}$$

In the above diagram,  $\text{id}_j, \text{id}_{j+1}$  are identity maps, and  $I, I_{\text{reg}}$  are inclusion maps.

Lemma 5.11 implies  $\lim_j H_*^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}^{a_j}) \cong \lim_j H_*^{\text{dR}}(\bar{\mathcal{L}}_k^{a_j})$ , then we obtain  $H_*^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}^a) \cong H_*^{\text{dR}}(\bar{\mathcal{L}}_k^a)$ , that is Lemma 5.8.

**Proof of Lemma 5.11.** Let  $\delta := a_{j+1}/a_j - 1$ . By Lemma 3.4 and Remark 3.5, there exists an integer  $D$  and a  $C^\infty$ -map  $F : M \times \mathbb{R}^D \rightarrow M$  such that

- For any  $z \in \mathbb{R}^D$ ,  $F_z : M \rightarrow M; x \mapsto F(x, z)$  is a diffeomorphism. Moreover, there holds  $|dF_z(v)| \leq (1 + \delta)|v|$  for any  $v \in TM$ .
- $F_{(0, \dots, 0)} = \text{id}_M$ .
- For any  $x \in M$ ,  $\mathbb{R}^D \rightarrow M; z \mapsto F(x, z)$  is a submersion.

Let us define  $\mathcal{F} : \bar{\mathcal{L}}_k^{a_j} \times \mathbb{R}^D \rightarrow \bar{\mathcal{L}}_k^{a_{j+1}}$  by

$$\mathcal{F}(\Gamma_0, \dots, \Gamma_k, z) := (F_z \circ \Gamma_0, \dots, F_z \circ \Gamma_k).$$

Then,  $(U \times \mathbb{R}^D, \mathcal{F} \circ (\varphi \times \text{id}_{\mathbb{R}^D})) \in \mathcal{P}(\bar{\mathcal{L}}_{k,\text{reg}}^{a_{j+1}})$  for any  $(U, \varphi) \in \mathcal{P}(\bar{\mathcal{L}}_k^{a_j})$ . Let us take  $\nu \in \mathcal{A}_c^D(\mathbb{R}^D)$  such that  $\int_{\mathbb{R}^D} \nu = 1$ . It is easy to see that

$$J : C_*^{\text{dR}}(\bar{\mathcal{L}}_k^{a_j}) \rightarrow C_*^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}^{a_{j+1}}); \quad [(U, \varphi, \omega)] \mapsto [(U \times \mathbb{R}^D, \mathcal{F} \circ (\varphi \times \text{id}_{\mathbb{R}^D}), \omega \times \nu)]$$

is a well-defined chain map. We show that  $J$  satisfies the requirement in Lemma 5.11.

To show that  $J \circ (\text{id}_j)_*$  and  $(I_{\text{reg}})_*$  are homotopic, we take  $a, b \in C^\infty(\mathbb{R}, [0, 1])$  so that

- $a(s) = 0$  for any  $s \leq 0$ , and  $a(s) = 1$  for any  $s \geq 1$ .
- $\text{supp } b$  is compact, and there exists  $\varepsilon > 0$  such that  $b(s) = 1$  for any  $s \in [-\varepsilon, 1 + \varepsilon]$ .

For any  $(U, \varphi) \in \mathcal{P}(\bar{\mathcal{L}}_{k,\text{reg}}^{a_j})$ , we define  $(U \times \mathbb{R}^D \times \mathbb{R}, \Phi) \in \mathcal{P}(\bar{\mathcal{L}}_k^{a_{j+1}})$  by

$$\Phi(u, z, s) := \mathcal{F}(\varphi(u), a(s)z).$$



Let us show that it is a plot of  $\bar{\mathcal{L}}_{k,\text{reg}}$ , namely  $e_i \circ \Phi : U \times \mathbb{R}^D \times \mathbb{R} \rightarrow M$  is a submersion for every  $0 \leq i \leq k$ . It is easy to check that  $e_i \circ \Phi(u, z, s) = F_{a(s)z} \circ e_i \circ \varphi(u)$ . Since  $(U, \varphi) \in \mathcal{P}(\bar{\mathcal{L}}_{k,\text{reg}})$ ,  $e_i \circ \varphi : U \rightarrow M$  is a submersion. On the other hand,  $F_{a(s)z}$  is a diffeomorphism on  $M$ . Thus,  $e_i \circ \Phi$  is a submersion. Hence  $(U \times \mathbb{R}^D \times \mathbb{R}, \Phi) \in \mathcal{P}(\bar{\mathcal{L}}_{k,\text{reg}}^{a_{j+1}})$ .

It is easy to see that the linear map

$$K : C_*^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}^{a_j}) \rightarrow C_{*+1}^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}^{a_{j+1}}); \quad [(U, \varphi, \omega)] \mapsto (-1)^{|\omega|+D}[(U \times \mathbb{R}^D \times \mathbb{R}, \Phi, \omega \times \nu \times b(s))]$$

is well-defined, and satisfies  $\partial K + K\partial = (I_{\text{reg}})_* - J \circ (\text{id}_j)_*$ .

Similar arguments show that  $(\text{id}_{j+1})_* \circ J$  is homotopic to  $I_*$ . The homotopy operator is given by exactly the same formula as  $K$ . This case is easier, since we do not have to care about the submersion condition.  $\square$

**5.4. Proof of Lemma 5.9.** Let us take strictly increasing sequences  $(a_j)_{j \geq 1}$ ,  $(E_j)_{j \geq 1}$  of positive real numbers, such that  $\lim_{j \rightarrow \infty} a_j = a$ ,  $\lim_{j \rightarrow \infty} E_j = \infty$ . We set

$$\mathcal{L}^{a_j, E_j} := \{\gamma \in \mathcal{L} \mid \text{len}(\gamma) < a_j, \mathcal{E}(\gamma) < E_j\}, \quad \mathcal{L}_k^{a_j, E_j} := \mathcal{L}_k \cap \mathcal{L}^{a_j, E_j} \times \Delta^k.$$

**Lemma 5.12.** *For every  $j \geq 1$ , there exists a smooth map  $J : \mathcal{L}^{a_j, E_j} \times \Delta^k \rightarrow \mathcal{L}_k^{a_{j+1}, E_{j+1}}$  such that the following diagram commutes up to smooth homotopy:*

$$\begin{array}{ccc} \mathcal{L}_k^{a_j, E_j} & \xrightarrow{\quad} & \mathcal{L}^{a_j, E_j} \times \Delta^k \\ \downarrow & \swarrow J & \downarrow \\ \mathcal{L}_k^{a_{j+1}, E_{j+1}} & \xrightarrow{\quad} & \mathcal{L}^{a_{j+1}, E_{j+1}} \times \Delta^k. \end{array}$$

All maps other than  $J$  are inclusion maps.

Assuming Lemma 5.12, we can prove Lemma 5.9 as

$$H_*^{\text{dR}}(\mathcal{L}_k^a) \cong \varinjlim_j H_*^{\text{dR}}(\mathcal{L}_k^{a_j, E_j}) \cong \varinjlim_j H_*^{\text{dR}}(\mathcal{L}^{a_j, E_j} \times \Delta^k) \cong H_*^{\text{dR}}(\mathcal{L}^a \times \Delta^k).$$

To prove Lemma 5.12, we need the following sublemma.

**Lemma 5.13.** *For any  $\delta > 0$ , there exists a  $C^\infty$ -map  $\mu : \Delta^k \times [0, 1] \rightarrow [0, 1]$  such that the following properties hold for any  $(t_1, \dots, t_k) \in \Delta^k$ .*

- (i):  $\mu(t_1, \dots, t_k, 0) = 0$ ,  $\mu(t_1, \dots, t_k, 1) = 1$ .
- (ii):  $\partial_\theta \mu(t_1, \dots, t_k, \theta) \in [0, 1 + \delta]$  for any  $\theta \in [0, 1]$ .
- (iii):  $|\mu(t_1, \dots, t_k, \theta) - \theta| \leq \delta$  for any  $\theta \in [0, 1]$ .
- (iv):  $\partial_\theta^m \mu(t_1, \dots, t_k, \theta) = 0$  for any integer  $m \geq 1$  and  $\theta \in \{0, t_1, \dots, t_k, 1\}$ .

**Proof.** We may assume  $\delta < 1$ . Let us take  $c \in (0, \delta/4(k+2))$ . We also take  $\kappa \in C^\infty(\mathbb{R}, [0, 1])$  such that  $\kappa^{(m)}(0) = 0$  for any integer  $m \geq 0$ , and  $\kappa(\theta) = 1$  if  $|\theta| \geq c$ . For

any  $t \in [0, 1]$ , we set  $\kappa_t(\theta) := \kappa(\theta - t)$ , and we define  $\nu, \tilde{\nu}, \mu \in C^\infty(\Delta^k \times [0, 1])$  by

$$\begin{aligned}\nu(t_1, \dots, t_k, \theta) &:= \kappa_0(\theta) \cdot \kappa_1(\theta) \cdot \prod_{1 \leq j \leq k} \kappa_{t_j}(\theta), \\ \tilde{\nu}(t_1, \dots, t_k, \theta) &:= \int_0^\theta \nu(t_1, \dots, t_k, \theta') d\theta', \\ \mu(t_1, \dots, t_k, \theta) &:= \tilde{\nu}(t_1, \dots, t_k, \theta) / \tilde{\nu}(t_1, \dots, t_k, 1).\end{aligned}$$

It is clear that  $\mu$  satisfies (i) and (iv). It is also easy to check  $\max\{0, \theta - \delta/2\} \leq \tilde{\nu}(t_1, \dots, t_k, \theta) \leq \theta$ , then (ii) and (iii) are verified as

$$\begin{aligned}\partial_\theta \mu(t_1, \dots, t_k, \theta) &\leq \nu(t_1, \dots, t_k, \theta) / (1 - \delta/2) \leq 1 + \delta, \\ \mu(t_1, \dots, t_k, \theta) &\geq \tilde{\nu}(t_1, \dots, t_k, \theta) \geq \theta - \delta/2, \\ \mu(t_1, \dots, t_k, \theta) &\leq \theta / (1 - \delta/2) \leq \theta(1 + \delta).\end{aligned}$$

□

**Proof of Lemma 5.12.** Let us fix  $E \in (E_j, E_{j+1})$ , and take  $\delta > 0$  so that

$$1 + \delta < (E/E_j)^{1/2}, \quad (E_j\delta)^{1/2} < r(\min\{a_{j+1}/a_j, (E_{j+1}/E)^{1/2}\} - 1).$$

For the definition of  $r(\cdots)$  in the second inequality, see Lemma 4.4.

We take  $\mu : \Delta^k \times [0, 1] \rightarrow [0, 1]$  as in Lemma 5.13. For any  $\gamma \in \mathcal{L}^{a_j, E_j}$  and  $(t_1, \dots, t_k) \in \Delta^k$ , we define  $\gamma_{t_1, \dots, t_k} \in \mathcal{L}$  by  $\gamma_{t_1, \dots, t_k}(\theta) := \gamma(\mu(t_1, \dots, t_k, \theta))$  (this is well-defined by (i)). Then  $\text{len}(\gamma_{t_1, \dots, t_k}) = \text{len}(\gamma)$ , and  $\mathcal{E}(\gamma_{t_1, \dots, t_k}) < E$  by (ii). By (iv), for any  $m \geq 1$  and  $\theta \in \{0, t_1, \dots, t_k\}$ , there holds  $\gamma_{t_1, \dots, t_k}^{(m)}(\theta) = 0$ . Therefore,  $(\gamma_{t_1, \dots, t_k}, t_1, \dots, t_k) \in \mathcal{L}_k^{a_j, E}$ .

Since  $\mathcal{L}_k^{a_j, E} \subset \mathcal{L}_k^{a_{j+1}, E_{j+1}}$ , one can define  $J$  by

$$J : \mathcal{L}^{a_j, E_j} \times \Delta^k \rightarrow \mathcal{L}_k^{a_{j+1}, E_{j+1}}; \quad (\gamma, t_1, \dots, t_k) \mapsto (\gamma_{t_1, \dots, t_k}, t_1, \dots, t_k).$$

By Lemma 5.13 (iii) and  $\mathcal{E}(\gamma) < E_j$ , there holds

$$\max_{\theta \in S^1} d(\gamma(\theta), \gamma_{t_1, \dots, t_k}(\theta)) \leq (E_j\delta)^{1/2}.$$

For any  $s \in [0, 1]$ , let  $\gamma_{s, t_1, \dots, t_k}(\theta) := F_s(\gamma(\theta), \gamma_{t_1, \dots, t_k}(\theta))$ . The map  $F_s$  is defined right before Lemma 4.4. Applying Lemma 4.4 for  $\min\{a_{j+1}/a_j, (E_{j+1}/E)^{1/2}\} - 1$ , we obtain  $\gamma_{s, t_1, \dots, t_k} \in \mathcal{L}^{a_{j+1}, E_{j+1}}$  for any  $s \in [0, 1]$ . If  $(\gamma, t_1, \dots, t_k) \in \mathcal{L}_k^{a_j, E_j}$ , there holds  $\gamma_{s, t_1, \dots, t_k}^{(m)}(\theta) = 0$  for any  $m \geq 1$ ,  $0 \leq s \leq 1$ , and  $\theta \in \{0, t_1, \dots, t_k\}$ . Thus,  $(\gamma_{s, t_1, \dots, t_k}, t_1, \dots, t_k) \in \mathcal{L}_k^{a_{j+1}, E_{j+1}}$  for any  $0 \leq s \leq 1$ .

Let us take  $\alpha \in C^\infty(\mathbb{R}, [0, 1])$  so that  $\alpha(s) = 0$  for  $s \leq 0$  and  $\alpha(s) = 1$  for  $s \geq 1$ . We define  $H : \mathcal{L}^{a_j, E_j} \times \Delta^k \times \mathbb{R} \rightarrow \mathcal{L}^{a_{j+1}, E_{j+1}} \times \Delta^k$  by

$$H(\gamma, t_1, \dots, t_k, s) := (\gamma_{\alpha(s), t_1, \dots, t_k}, t_1, \dots, t_k).$$

Obviously, this is a smooth homotopy between  $J$  and the inclusion map  $\mathcal{L}^{a_j, E_j} \times \Delta^k \rightarrow \mathcal{L}^{a_{j+1}, E_{j+1}} \times \Delta^k$ . Finally, the restriction of  $H$  to  $\mathcal{L}_k^{a_j, E_j} \times \mathbb{R}$  is a smooth homotopy between  $J|_{\mathcal{L}_k^{a_j, E_j}}$  and the inclusion map  $\mathcal{L}_k^{a_j, E_j} \rightarrow \mathcal{L}_k^{a_{j+1}, E_{j+1}}$ . □

**5.5. Proof of Lemma 5.10.** It is easy to see that the map  $\mathcal{L}_k^a \rightarrow \bar{\mathcal{L}}_k^a$  is smooth, thus it is enough to show that the map induces an isomorphism on  $H_*^{\text{dR}}$ . The proof consists of three steps.

**Step 1.** We identify  $\bar{\mathcal{L}}_k$  with the set consists of tuples  $(\gamma, t_1, \dots, t_k, T)$  (see Remark 5.5). Setting  $p : [0, 1] \rightarrow \mathbb{R}/\mathbb{Z}$  by  $p(\theta) := [\theta]$ , the map  $\mathcal{L}_k^a \rightarrow \bar{\mathcal{L}}_k^a$  is given by  $(\gamma, t_1, \dots, t_k) \mapsto (\gamma \circ p, t_1, \dots, t_k, 1)$ . The image of this map is contained in

$$\bar{\mathcal{L}}_{k,T>0}^a := \{(\gamma, t_1, \dots, t_k, T) \in \bar{\mathcal{L}}_k^a \mid T > 0\}.$$

Now,  $\mathcal{L}_k^a \rightarrow \bar{\mathcal{L}}_{k,T>0}^a$  induces an isomorphism on  $H_*^{\text{dR}}$ , since

$$\bar{\mathcal{L}}_{k,T>0}^a \rightarrow \mathcal{L}_k^a; \quad (\gamma, t_1, \dots, t_k, T) \mapsto (\gamma_T, t_1/T, \dots, t_k/T)$$

is its smooth homotopy inverse, where  $\gamma_T(\theta) := \gamma(T\theta)$ . Therefore, it is enough to show that the inclusion map  $\bar{\mathcal{L}}_{k,T>0}^a \rightarrow \bar{\mathcal{L}}_k^a$  gives an isomorphism on  $H_*^{\text{dR}}$ .

**Step 2.** Let  $\mathcal{E}(\gamma, T) := \int_0^T |\dot{\gamma}|^2$ . For any  $E > 0$ , let us set

$$\bar{\mathcal{L}}_k^{a,E} := \{(\gamma, t_1, \dots, t_k, T) \in \bar{\mathcal{L}}_k^a \mid \mathcal{E}(\gamma, T) < E\}, \quad \bar{\mathcal{L}}_{k,T>0}^{a,E} := \bar{\mathcal{L}}_{k,T>0}^a \cap \bar{\mathcal{L}}_k^{a,E}.$$

Since  $\mathcal{E}$  is smooth as a function on  $\bar{\mathcal{L}}_k^a$ , we obtain isomorphisms  $H_*^{\text{dR}}(\bar{\mathcal{L}}_k^a) \cong \varinjlim_{E \rightarrow \infty} H_*^{\text{dR}}(\bar{\mathcal{L}}_k^{a,E})$ ,

$H_*^{\text{dR}}(\bar{\mathcal{L}}_{k,T>0}^a) \cong \varinjlim_{E \rightarrow \infty} H_*^{\text{dR}}(\bar{\mathcal{L}}_{k,T>0}^{a,E})$ . Thus, it is enough to show that the inclusion map  $\bar{\mathcal{L}}_{k,T>0}^{a,E} \rightarrow \bar{\mathcal{L}}_k^{a,E}$  induces an isomorphism on  $H_*^{\text{dR}}$  for every  $E > 0$ .

**Step 3.** For any  $E, \delta > 0$ , let us set

$$\begin{aligned} \bar{\mathcal{L}}_{k,\delta}^{a,E} &:= \{(\gamma, t_1, \dots, t_k, T) \in \bar{\mathcal{L}}_k^{a,E} \mid \gamma \text{ is a constant loop if } T < \delta\}, \\ \bar{\mathcal{L}}_{k,T>0,\delta}^{a,E} &:= \bar{\mathcal{L}}_{k,T>0}^{a,E} \cap \bar{\mathcal{L}}_{k,\delta}^{a,E}. \end{aligned}$$

Consider the following commutative diagram, where all maps are inclusion maps:

$$\begin{array}{ccc} \bar{\mathcal{L}}_{k,T>0,\delta}^{a,E} & \xrightarrow{j_3} & \bar{\mathcal{L}}_{k,\delta}^{a,E} \\ j_1 \downarrow & & \downarrow j_2 \\ \bar{\mathcal{L}}_{k,T>0}^{a,E} & \xrightarrow{j_4} & \bar{\mathcal{L}}_k^{a,E}. \end{array}$$

Then, it is easy to check the following claims:

- There exists  $\delta(E) > 0$ , depending only on  $E$ , such that if  $\delta < \delta(E)$  then  $j_1, j_2$  induce isomorphisms on  $H_*^{\text{dR}}$ . This is because one can define homotopy inverses of  $j_1$  and  $j_2$  by contracting a loop  $(\gamma, T)$  to the constant loop at  $\gamma(0)$  whenever  $T < \delta$  (notice that  $\text{len}(\gamma) \leq \sqrt{ET}$ ).
- For any  $\delta > 0$ ,  $j_3$  induces an isomorphism on  $H_*^{\text{dR}}$ , since its homotopy inverse is given by  $(\gamma, t_1, \dots, t_k, T) \mapsto (\gamma_T, t_1, \dots, t_k, \nu(T))$ , such that
  - $\nu(T) \geq T$  for any  $T \geq 0$ ,  $\nu(T) = T$  for any  $T \geq \delta/2$ , and  $\nu(0) > 0$ .
  - $\gamma_T$  is defined as  $\gamma_T := \begin{cases} \gamma & (T \geq \delta) \\ \text{constant loop at } \gamma(0) & (T < \delta). \end{cases}$

Therefore, we can conclude that  $j_4$  induces an isomorphism on  $H_*^{\text{dR}}$ . This completes the proof of Lemma 5.10.

## 6. THE CHAIN COMPLEX $C_*^{\mathcal{L}^M}$

The first goal of this section is to define the chain complex  $C_*^{\mathcal{L}^M}$ , which is our chain model of the free loop space  $\mathcal{L}^M$ . The second goal is to prove the results presented in Section 1.5 assuming Theorem 6.7, which is a purely algebraic result on dg operads. The proof of Theorem 6.7 occupies the rest of this paper (Sections 7–11).

After some algebraic preliminaries in Section 6.1, we define the chain complex  $C_*^{\mathcal{L}^M}$  in Section 6.2. In Section 6.3, we establish the relation between  $C_*^{\mathcal{L}^M}$  and the Hochschild complex of  $\mathcal{A}_M$ . In Section 6.4, we state Theorem 6.7, and prove most results presented in Section 1.5, except that the isomorphism  $\mathbb{H}_*(\mathcal{L}^M) \cong H_*(C^{\mathcal{L}^M})$  preserves the BV structures. We prove this result in Sections 6.5–6.8.

**6.1. Algebraic preliminaries.** A double complex  $C$  consists of a sequence  $(C(k)_*)_{k \geq 0}$  of chain complexes and anti-chain maps  $\delta_k : C(k-1)_* \rightarrow C(k)_*$  for each  $k \geq 1$ , such that  $\delta_{k+1} \circ \delta_k = 0$  for any  $k \geq 1$ . We denote  $\delta_k$  as  $\delta_k^C$  if necessary. For any double complexes  $C$  and  $D$ , a morphism  $\varphi : C \rightarrow D$  is a sequence  $\varphi = (\varphi(k))_{k \geq 0}$  such that,  $\varphi(k) : C(k)_* \rightarrow D(k)_*$  is a chain map and  $\delta_k^D \circ \varphi(k-1) = \varphi(k) \circ \delta_k^C$  for every  $k \geq 1$ .

For any double complex  $C$ , we define a chain complex  $(\tilde{C}, \tilde{\partial})$  by

$$\tilde{C}_* := \prod_{k=0}^{\infty} C(k)_{*+k}, \quad (\tilde{\partial}x)_k := \partial x_k + \delta_k(x_{k-1}),$$

which we call the *total complex* of  $C$ . A morphism  $\varphi : C \rightarrow D$  of double complexes induces a chain map  $\tilde{\varphi} : \tilde{C} \rightarrow \tilde{D}$ ;  $(x_k)_{k \geq 0} \mapsto (\varphi(k)(x_k))_{k \geq 0}$ .

**Lemma 6.1.** *Let  $C = (C(k))_{k \geq 0}$  be a double complex. If the sequence*

$$0 \longrightarrow H_q(C(0)) \xrightarrow{H_q(\delta_1)} H_q(C(1)) \xrightarrow{H_q(\delta_2)} H_q(C(2)) \xrightarrow{H_q(\delta_3)} \cdots$$

*is exact for every  $q \in \mathbb{Z}$ , then the total complex  $\tilde{C}$  is acyclic.*

**Proof.** For any  $l \geq 0$ , let  $F_l \tilde{C}_* := \prod_{k \geq l} C(k)_{*+k}$ . Then,  $(F_l \tilde{C})_{l \geq 0}$  is a decreasing filtration on  $\tilde{C}$  which is complete, i.e.  $\tilde{C} \cong \varprojlim_{l \rightarrow \infty} \tilde{C}/F_l \tilde{C}$ . Let us consider the spectral sequence of this filtered complex. Then, the assumption implies that all  $E^2$ -terms vanish. Now, the convergence theorem 5.5.10 (2) in [32] pp.139 shows  $H_*(\tilde{C}) = 0$ .  $\square$

**Lemma 6.2.** *Let  $\varphi : C \rightarrow D$  be a morphism of double complexes. Suppose that  $\varphi(k) : C(k)_* \rightarrow D(k)_*$  is a quasi-isomorphism for every  $k \geq 0$ . Then, the chain map  $\tilde{\varphi} : \tilde{C} \rightarrow \tilde{D}$  is a quasi-isomorphism.*

**Proof.** Let us consider the filtrations  $(F_l \tilde{C})_{l \geq 0}$  and  $(F_l \tilde{D})_{l \geq 0}$  as in the proof of the previous lemma. Then,  $\tilde{\varphi} : \tilde{C} \rightarrow \tilde{D}$  induces a morphism of the spectral sequences, and the

assumption implies that it induces isomorphisms on  $E^1$ -terms. Then, the comparison theorem 5.5.11 in [32] pp. 141 shows that  $H_*(\tilde{\varphi}) : H_*(\tilde{C}) \rightarrow H_*(\tilde{D})$  is an isomorphism.  $\square$

Let  $C = (C(k)_*)_{k \geq 0}$  be any cosimplicial chain complex. For any  $k \geq 1$ , let us define  $\delta_k : C(k-1)_* \rightarrow C(k)_*$  by  $\delta_k(x) := (-1)^{|x|+k-1} \sum_{i=0}^k (-1)^i \delta_{k,i}(x)$ . Then,  $C$  is a double complex.

**Example 6.3.** For any dga algebra  $A$ , the endomorphism operad  $\text{End}(A) := (\text{Hom}(A^{\otimes k}, A))_{k \geq 0}$  has a multiplication  $\mu \in \text{Hom}_0(A^{\otimes 2}, A)$  and a unit  $\varepsilon \in \text{Hom}_0(\bar{\mathbb{R}}, A)$ , defined by  $\mu(a_1 \otimes a_2) := a_1 a_2$  and  $\varepsilon(1) := 1_A$ . In particular,  $\text{End}(A)$  has the structure of a cosimplicial chain complex. The total complex  $\widetilde{\text{End}(A)}$  is isomorphic to the Hochschild cochain complex  $C^*(A, A)$ .

As a consequence of Lemma 6.1, we obtain the next lemma.

**Lemma 6.4.** *Let  $C = (C(k)_*)_{k \geq 0}$  be a cosimplicial chain complex, and suppose that the chain map  $C(k)_* \rightarrow C(0)_*$ , which is induced by the cosimplicial structure, is a quasi-isomorphism for any  $k \geq 0$ . Then, the projection map  $\text{pr}_0 : \tilde{C}_* \rightarrow C(0)_*; (x_k)_{k \geq 0} \mapsto x_0$  is a quasi-isomorphism.*

**Proof.** Since  $\text{pr}_0$  is surjective, it is enough to show that  $\ker(\text{pr}_0) = \prod_{k=1}^{\infty} C(k)_{*+k}$  is acyclic.

The assumption shows that, for any  $q \in \mathbb{Z}$  the sequence

$$0 \longrightarrow H_q(C(1)) \xrightarrow{H_q(\delta_2)} H_q(C(2)) \xrightarrow{H_q(\delta_3)} H_q(C(3)) \xrightarrow{H_q(\delta_4)} \cdots$$

is exact, thus Lemma 6.1 shows that  $\ker(\text{pr}_0)$  is acyclic.  $\square$

**6.2. Definition of  $C_*^{\mathcal{L}^M}$ .** Let  $M$  be a closed, oriented Riemannian manifold of dimension  $d$ . We define the chain complex  $C_*^{\mathcal{L}^M}$  to be the total complex of the cosimplicial chain complex  $(C_{*+d}^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}))_{k \geq 0}$ . Explicitly,

$$C_*^{\mathcal{L}^M} := \prod_{k=0}^{\infty} C_{*+d+k}^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}),$$

$$\partial : C_*^{\mathcal{L}^M} \rightarrow C_{*-1}^{\mathcal{L}^M}; \quad (x_k)_{k \geq 0} \mapsto (\partial x_k)_{k \geq 0} + (\delta_k x_{k-1})_{k \geq 1}.$$

For any  $a \in (0, \infty]$ , we define a subcomplex  $F^a C_*^{\mathcal{L}^M} \subset C_*^{\mathcal{L}^M}$  by  $F^a C_*^{\mathcal{L}^M} := \prod_{k=0}^{\infty} C_{*+d+k}^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}^a)$ .

Then,  $(F^a C_*^{\mathcal{L}^M})_{a \in (0, \infty]}$  is the length filtration on  $C_*^{\mathcal{L}^M}$ .

We are going to show that, there exists an isomorphism  $\mathbb{H}_*(\mathcal{L}^a M) \cong H_*(F^a C_*^{\mathcal{L}^M})$  for each  $a \in (0, \infty]$ , such that Proposition 1.10 (i) holds. In particular, when  $a = \infty$  we obtain an isomorphism  $\mathbb{H}_*(\mathcal{L} M) \cong H_*(C_*^{\mathcal{L}^M})$ . Let us abbreviate  $\mathcal{L}^a M$  by  $\mathcal{L}^a$ , as usual.

Let us recall the zig-zag (8) of quasi-isomorphisms from Section 5.2:

$$C_{*+d}^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}^a) \longrightarrow C_{*+d}^{\text{dR}}(\bar{\mathcal{L}}_k^a) \longleftarrow C_{*+d}^{\text{dR}}(\mathcal{L}_k^a) \longrightarrow C_{*+d}^{\text{dR}}(\mathcal{L}^a \times \Delta^k).$$

It extends to a zig-zag of quasi-isomorphisms of cocylic chain complexes. Let  $F^a C_*^{\mathcal{L}\Delta}$  denote the total complex of the cosimplicial chain complex  $(C_{*+d}^{\text{dR}}(\mathcal{L}^a \times \Delta^k))_{k \geq 0}$ . Then, Lemma 6.2 implies the isomorphism  $H_*(F^a C^{\mathcal{L}M}) \cong H_*(F^a C^{\mathcal{L}\Delta})$ .

For every  $k \geq 0$  the projection  $\mathcal{L}^a \times \Delta^k \rightarrow \mathcal{L}^a$  induces a quasi-isomorphism on  $C_*^{\text{dR}}$ . Then, Lemma 6.4 shows that  $\text{pr}_0 : F^a C_*^{\mathcal{L}\Delta} \rightarrow C_{*+d}^{\text{dR}}(\mathcal{L}^a)$  is a quasi-isomorphism, thus  $H_*(F^a C^{\mathcal{L}\Delta}) \cong H_{*+d}^{\text{dR}}(\mathcal{L}^a)$ . The next lemma is confirmed by short computations.

**Lemma 6.5.** *Let  $u = (u_k)_{k \geq 0}$  be as in Lemma 2.12 (i). Then,*

$$E_u : C_{*+d}^{\text{dR}}(\mathcal{L}^a) \rightarrow F^a C_*^{\mathcal{L}\Delta}; \quad x \mapsto ((-1)^{k(2d+k+1)/2} x \times u_k)_{k \geq 0}$$

*is a chain map such that  $\text{pr}_0 \circ E_u = \text{id}_{C^{\text{dR}}(\mathcal{L}^a)}$ .*

On the other hand, we have the isomorphism  $\mathbb{H}_*(\mathcal{L}^a) \cong H_{*+d}^{\text{dR}}(\mathcal{L}^a)$  by Theorem 4.1. In conclusion, for every  $a \in (0, \infty]$ , we obtain a sequence of isomorphisms

$$\mathbb{H}_*(\mathcal{L}^a) \cong H_{*+d}^{\text{dR}}(\mathcal{L}^a) \cong H_*(F^a C^{\mathcal{L}\Delta}) \cong H_*(F^a C^{\mathcal{L}M}).$$

In particular, we obtain  $\mathbb{H}_*(\mathcal{L}) \cong H_*(C^{\mathcal{L}M})$ .

To define a chain map  $\iota_M : (\mathcal{A}_M)_* \rightarrow C_*^{\mathcal{L}M}$ , we take  $M' \in \mathcal{U}$  and an orientation-preserving diffeomorphism  $\varphi : M' \rightarrow M$ . Let us recall the map  $i_0 : M \rightarrow \tilde{\mathcal{L}}_0$ ;  $p \mapsto c_p$ , and define  $\iota_M$  by

$$(\iota_M(\omega))_k := \begin{cases} [(M', i_0 \circ \varphi, \varphi^* \omega)] & (k = 0), \\ 0 & (k \geq 1). \end{cases}$$

It is easy to check that this is well-defined (i.e. not depend on  $M'$  and  $\varphi$ ), and the diagram (4) commutes.

**6.3. Relation to the Hochschild complex of  $\mathcal{A}_M$ .** For any  $k \geq 0$ ,  $(U, \varphi) \in \mathcal{P}(\tilde{\mathcal{L}}_{k,\text{reg}})$  and  $j = 0, \dots, k$ , let  $\varphi_j := e_j \circ \varphi$ . By definition,  $\varphi_j$  are submersions for all  $j$ . We define a chain map  $J_k : C_{*+d}^{\text{dR}}(\tilde{\mathcal{L}}_{k,\text{reg}}) \rightarrow \text{Hom}_*(\mathcal{A}_M^{\otimes k}, \mathcal{A}_M)$  by

$$J_k([(U, \varphi, \omega)]) (\eta_1 \otimes \dots \otimes \eta_k) := (-1)^{(\dim U - d)(|\eta_1| + \dots + |\eta_k|)} (\varphi_0)_! (\omega \wedge \varphi_1^* \eta_1 \wedge \dots \wedge \varphi_k^* \eta_k).$$

Recall that we defined nonsymmetric dg operads with multiplications and units

$$\mathcal{C}\mathcal{L} = (C_{*+d}^{\text{dR}}(\tilde{\mathcal{L}}_{k,\text{reg}}))_{k \geq 0}, \quad \text{End}(\mathcal{A}_M) = (\text{Hom}_*(\mathcal{A}_M^{\otimes k}, \mathcal{A}_M))_{k \geq 0}.$$

$(J_k)_{k \geq 0} : \mathcal{C}\mathcal{L} \rightarrow \text{End}(\mathcal{A}_M)$  is a morphism of nonsymmetric dg operads preserving multiplications and units.

In general, if  $\varphi : \mathcal{O}_0 \rightarrow \mathcal{O}_1$  is a morphism of nonsymmetric dg operads preserving multiplications and units, then  $\varphi$  is a morphism of cosimplicial chain complexes, thus induces a chain map  $\tilde{\varphi} : \tilde{\mathcal{O}}_0 \rightarrow \tilde{\mathcal{O}}_1$ . In particular, we obtain a chain map  $J : C_*^{\mathcal{L}M} \rightarrow C^*(\mathcal{A}_M, \mathcal{A}_M)$ . We are going to show the following lemma (see Theorem 1.5 (iii)-(c)).

**Lemma 6.6.**  $H_*(J) \circ \Phi : \mathbb{H}_*(\mathcal{L}M) \rightarrow H^*(\mathcal{A}_M, \mathcal{A}_M)$  is equal to the map (2).

**Proof.** For any  $j = 0, \dots, k$ , we define  $e_j : \mathcal{L} \times \Delta^k \rightarrow M$  by

$$e_j(\gamma, t_1, \dots, t_k) := \begin{cases} \gamma(0) & (j = 0) \\ \gamma(t_j) & (1 \leq j \leq k). \end{cases}$$

For any  $(U, \varphi) \in \mathcal{P}(\mathcal{L} \times \Delta^k)$ , we set  $\varphi_j := e_j \circ \varphi$ . Let us define a chain map  $J'_k : C_*^{\text{dR}}(\mathcal{L} \times \Delta^k) \rightarrow \text{Hom}(\mathcal{A}_M^{\otimes k}, \mathcal{A}_M^\vee[d])$  by

$$J'_k[(U, \varphi, \omega)](\eta_1 \otimes \cdots \otimes \eta_k)(\eta_0) := (-1)^{(\dim U - d)(|\eta_0| + |\eta_1| + \cdots + |\eta_k|)} \int_U \omega \wedge \varphi_1^* \eta_1 \wedge \cdots \wedge \varphi_k^* \eta_k \wedge \varphi_0^* \eta_0.$$

Then, one can define a chain map  $J' = (J'_k)_{k \geq 0} : C_*^{\mathcal{L}\Delta} \rightarrow C^*(\mathcal{A}_M, \mathcal{A}_M^\vee[d])$ . As is obvious from the construction, the following diagram commutes:

$$\begin{array}{ccc} H_*(C^{\mathcal{L}M}) & \xrightarrow{H_*(J)} & H^*(\mathcal{A}_M, \mathcal{A}_M) \\ \cong \downarrow & & \downarrow \cong \\ H_*(C^{\mathcal{L}\Delta}) & \xrightarrow{H_*(J')} & H^*(\mathcal{A}_M, \mathcal{A}_M^\vee[d]). \end{array}$$

Let us consider the chain map  $I : C_{*+d}^{\text{sm}}(\mathcal{L}) \rightarrow C^*(\mathcal{A}_M, \mathcal{A}_M^\vee[d])$  in (1). We need to show that  $H_*(J')$  corresponds to  $H_*(I)$  via the isomorphism  $H_{*+d}^{\text{sm}}(\mathcal{L}) \cong H_*(C^{\mathcal{L}\Delta})$ .

Let us take  $u = (u_k)_{k \geq 0}$  as in Lemma 2.12 (i), and consider chain maps

$$\iota^u(\mathcal{L})_* : C_*^{\text{sm}}(\mathcal{L}) \rightarrow C_*^{\text{dR}}(\mathcal{L}), \quad E_u : C_{*+d}^{\text{dR}}(\mathcal{L}) \rightarrow C_*^{\mathcal{L}\Delta}.$$

The first map is defined right after Lemma 2.12, and the second map is defined in Lemma 6.5. Then, it is enough to show that the following diagram commutes:

$$\begin{array}{ccccc} H_{*+d}^{\text{sm}}(\mathcal{L}) & \xrightarrow[\cong]{H_*(I_u)} & H_{*+d}^{\text{dR}}(\mathcal{L}) & \xrightarrow[\cong]{H_*(E_u)} & H_*(C^{\mathcal{L}\Delta}) \\ & \searrow H_*(I) & & \swarrow H_*(J') & \\ & & H^*(\mathcal{A}_M, \mathcal{A}_M^\vee[d]) & & \end{array}$$

The commutativity of this diagram follows from Lemma 2.14 and short computations.  $\square$

**6.4. Algebraic structures on  $C_*^{\mathcal{L}M}$ .** First we state the following result, which we prove in Sections 7–11.

**Theorem 6.7.** *There exists a dg operad  $f\tilde{\Lambda}$  and its suboperad  $\tilde{\Lambda}$  with the following conditions.*

- (i): *There exist isomorphisms of graded operads  $H_*(f\tilde{\Lambda}) \cong \mathcal{BV}$  and  $H_*(\tilde{\Lambda}) \cong \mathcal{G}$ , such that the following diagram commutes:*

$$\begin{array}{ccc} H_*(\tilde{\Lambda}) & \longrightarrow & H_*(f\tilde{\Lambda}) \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{G} & \longrightarrow & \mathcal{BV}. \end{array}$$

- (ii): *Let  $\mathcal{O}$  be any nonsymmetric dg operad, with a multiplication  $\mu \in \mathcal{O}(2)_0$  and a unit  $\varepsilon \in \mathcal{O}(0)_0$ . Then,  $\tilde{\mathcal{O}}$  has a dg  $\tilde{\Lambda}$ -algebra structure, in particular  $H_*(\tilde{\mathcal{O}})$  has the Gerstenhaber algebra structure. The operators  $\bullet$  and  $\{, \}$  are defined in chain*

level by the following formulas, where  $x = (x_l)_{l \geq 0}, y = (y_m)_{m \geq 0} \in \tilde{\mathcal{O}}_*$ :

$$\begin{aligned}(x \bullet y)_k &:= \sum_{l+m=k} (-1)^{l(|y|+m)} (\mu \circ_1 x_l) \circ_{l+1} y_m, \\ (x * y)_k &:= \sum_{\substack{l+m=k+1 \\ 1 \leq i \leq l}} (-1)^{|y|(l+1)+m(i+1)+l+i} x_l \circ_i y_m, \\ \{x, y\} &:= x * y - (-1)^{(|x|+1)(|y|+1)} y * x.\end{aligned}$$

These operators induce dga and dg Lie algebra structures on  $\tilde{\mathcal{O}}$ . Finally, for any morphism  $\mathcal{O}_0 \rightarrow \mathcal{O}_1$  of nonsymmetric dg operads which preserves multiplications and units, the chain map  $\tilde{\mathcal{O}}_0 \rightarrow \tilde{\mathcal{O}}_1$  is a morphism of dg  $\tilde{\Lambda}$ -algebras.

(iii): Let  $\mathcal{O}$  be as (ii), and assume that  $\mathcal{O}$  has a cyclic structure  $(\tau_k)_{k \geq 0}$  such that  $\tau_2(\mu) = \mu$ . Then,  $\tilde{\mathcal{O}}$  has a dg  $f\tilde{\Lambda}$ -algebra structure, which extends the dg  $\tilde{\Lambda}$ -algebra structure in (ii). In particular,  $H_*(\tilde{\mathcal{O}})$  has the BV algebra structure. The operator  $\Delta$  is defined at chain level by the following formula:

$$(\Delta x)_k = \sum_{i=0}^k (-1)^{ki+|x|+1} (\tau_{k+1}^{k+1-i} x_{k+1}) \circ_{i+1} \varepsilon.$$

$\Delta$  is an anti-chain map on  $\tilde{\mathcal{O}}$ .

**Remark 6.8.** There are several results similar to Theorem 6.7 in the literature, e.g. [21] Theorem 5.17, [31] Theorems A and B. Actually, [31] considers operads with Maurer-Cartan elements, which are generalizations of operads with multiplications. However, for the following reasons we cannot directly apply results in [31] in our argument:

- [31] assumes  $\mathcal{O}(0) = 0$ , which we cannot assume in our argument.
- In Theorem B in [31], actions of operads are defined only for *normalized* chains (see Section 3.1 [31]), while we would like to consider arbitrary chains.

Therefore, we give a self-contained proof of Theorem 6.7 in Sections 7–11.

As a consequence of Theorem 6.7, we obtain the following:

- For any dga algebra  $A$ , the Hochschild complex  $C^*(A, A)$  has a dg  $\tilde{\Lambda}$ -algebra structure; apply Theorem 6.7 (ii) for  $\mathcal{O} = \text{End}(A)$ .
- For any closed, oriented  $C^\infty$ -manifold  $M$ , the chain complex  $C_*^{\mathcal{L}^M}$  has a dg  $f\tilde{\Lambda}$ -algebra structure; apply Theorem 6.7 (iii) for  $\mathcal{O} = \mathcal{CL} = (C_{*+d}^{\text{dR}}(\bar{\mathcal{L}}_{k, \text{reg}}))_{k \geq 0}$ . Also, this  $f\tilde{\Lambda}$ -algebra structure on  $C_*^{\mathcal{L}^M}$  preserves the length filtration; see Remark 9.5.
- The chain map  $J : C_*^{\mathcal{L}^M} \rightarrow C^*(\mathcal{A}_M, \mathcal{A}_M)$  (see the previous subsection) is a map of dg  $\tilde{\Lambda}$ -algebras; since  $(J_k)_{k \geq 0} : \mathcal{CL} \rightarrow \text{End}(\mathcal{A}_M)$  is a morphism of nonsymmetric dg operads preserving multiplications and units, we can apply Theorem 6.7 (ii).
- The chain map  $\iota_M : (\mathcal{A}_M)_* \rightarrow C_*^{\mathcal{L}^M}$  satisfies (3); this follows from explicit formulas of  $\bullet$  and  $\{, \}$  in Theorem 6.7 (ii).

Now, we have verified most results presented in Section 1.5. The only statement we have not verified is that the isomorphism  $\Phi : \mathbb{H}_*(\mathcal{LM}) \cong H_*(C^{\mathcal{L}^M})$  preserves the BV structures, i.e. the operators  $\bullet$  and  $\Delta$  are preserved by this isomorphism. In the rest



of this section, we prove that  $\Phi$  preserves the operator  $\Delta$  (Section 6.5) and  $\bullet$  (Sections 6.6–6.8). The arguments in Sections 6.6–6.8 are less detailed than the other parts of this paper.

**6.5. Rotation operator  $\Delta$ .** The aim of this subsection is to show that the isomorphism  $\Phi : \mathbb{H}_*(\mathcal{L}M) \cong H_*(C^{\mathcal{L}M})$  preserves the rotation operator  $\Delta$ .

For any cocyclic chain complex  $C = (C(k)_*)_{k \geq 0}$  and its total complex  $\tilde{C}$ , let us define an anti-chain map  $\Delta : \tilde{C}_* \rightarrow \tilde{C}_{*+1}$  by

$$(\Delta x)_k = \sum_{i=0}^k (-1)^{ki+|x|+1} \sigma_{k,i} \tau_{k+1}^{k+1-i}(x_{k+1}) \quad (k \geq 0).$$

This is a generalization of the definition of  $\Delta$  in Theorem 6.7 (iii). In particular, we can define  $\Delta$  on  $C_*^{\mathcal{L}\Delta} = \prod_{k \geq 0} C_{*+d+k}^{\text{dR}}(\mathcal{L} \times \Delta^k)$  so that the isomorphism  $H_*(C^{\mathcal{L}M}) \cong H_*(C^{\mathcal{L}\Delta})$

preserves  $\Delta$ . Since we already proved that the isomorphism  $H_*^{\text{dR}}(\mathcal{L}) \cong H_*(\mathcal{L})$  preserves  $\Delta$  (Corollary 4.2), it is enough to show that the isomorphism  $H_{*+d}^{\text{dR}}(\mathcal{L}) \cong H_*(C^{\mathcal{L}\Delta})$  preserves  $\Delta$ .

The isomorphism  $H_{*+d}^{\text{dR}}(\mathcal{L}) \cong H_*(C^{\mathcal{L}\Delta})$  is induced by  $\text{pr}_0 : C_*^{\mathcal{L}\Delta} \rightarrow C_{*+d}^{\text{dR}}(\mathcal{L})$ , and its inverse is induced by  $E_u$  (Lemma 6.5), where  $u = (u_k)_{k \geq 0}$  is as in Lemma 2.12 (i). Therefore, it is enough to prove  $H_*(\text{pr}_0) \circ \Delta \circ H_*(E_u) = \Delta$  on  $H_{*+d}^{\text{dR}}(\mathcal{L})$ .

It is easy to check that  $\text{pr}_0 \circ \Delta \circ E_u(x) = (-1)^{|x|+1} \sigma_1 \tau_1(x \times u_1)$  for any  $x \in C_{*+d}^{\text{dR}}(\mathcal{L})$ . The chain map  $\sigma_1 \tau_1 : C_*^{\text{dR}}(\mathcal{L} \times \Delta^1) \rightarrow C_*^{\text{dR}}(\mathcal{L})$  is induced by

$$\mathcal{L} \times \Delta^1 \xrightarrow{\text{id}_{\mathcal{L}} \times p} \mathcal{L} \times S^1 \longrightarrow S^1 \times \mathcal{L} \xrightarrow{r} \mathcal{L},$$

where  $p : \Delta^1 \rightarrow S^1$  is defined by  $p(\theta) := [\theta]$ , the second map is the inversion, and the last map  $r$  is the rotation. Therefore  $\sigma_1 \tau_1(x \times u_1) = (-1)^{|x|} r_*(p_*(u_1) \times x)$  for any  $x \in C_{*+d}^{\text{dR}}(\mathcal{L})$ . Since  $p_*(u_1) \in C_1^{\text{dR}}(S^1)$  is a cycle which represents  $[S^1]$ , for any cycle  $x$  we obtain  $H_*(\text{pr}_0) \circ \Delta \circ H_*(E_u)([x]) = -H_*(r)([S^1] \times [x])$ . This completes the proof.

**6.6. The loop product  $\bullet$ .** We are going to show that the isomorphism  $\Phi : \mathbb{H}_*(\mathcal{L}M) \cong H_*(C^{\mathcal{L}M})$  preserves the operator  $\bullet$ . In this subsection, we reduce the proof to Lemma 6.10, which is proved in the next two subsections.

Let us consider the concatenation map

$$c : \bar{\mathcal{L}}_0 \times_{e_0} \bar{\mathcal{L}}_0 \rightarrow \bar{\mathcal{L}}_0; \quad ((\gamma_0, T_0), (\gamma_1, T_1)) \mapsto (\gamma_0 * \gamma_1, T_0 + T_1),$$

and define a chain map  $\bullet_0 : C_{*+d}^{\text{dR}}(\bar{\mathcal{L}}_{0,\text{reg}})^{\otimes 2} \rightarrow C_{*+d}^{\text{dR}}(\bar{\mathcal{L}}_{0,\text{reg}})$  by  $a \bullet_0 b := c_*(a \times_{e_0} b)$ . Then,  $\text{pr}_0 : C_*^{\mathcal{L}M} \rightarrow C_{*+d}^{\text{dR}}(\bar{\mathcal{L}}_{0,\text{reg}}); (x_k)_{k \geq 0} \mapsto x_0$  intertwines the operators  $\bullet$  and  $\bullet_0$ :

$$(x \bullet y)_0 = (\mu \circ_1 x_0) \circ_1 y_0 = c_*(x_0 \times_{e_0} y_0) = x_0 \bullet_0 y_0.$$

Therefore, it is enough to prove the next proposition.

**Proposition 6.9.** *The isomorphism  $H_{*+d}^{\text{dR}}(\bar{\mathcal{L}}_{0,\text{reg}}) \cong \mathbb{H}_*(\mathcal{L})$  intertwines the operator  $\bullet_0$  on  $H_{*+d}^{\text{dR}}(\bar{\mathcal{L}}_{0,\text{reg}})$  and the loop product  $\bullet$  on  $\mathbb{H}_*(\mathcal{L})$ .*

Let us define  $e : \mathcal{L} \rightarrow M$  by  $e(\gamma) := \gamma(0)$ . A key step in the definition of the loop product on  $\mathbb{H}_*(\mathcal{L})$  is to define the fiber product

$$H_*(\mathcal{L})^{\otimes 2} \rightarrow H_{*-d}(\mathcal{L} \times_e \mathcal{L})$$

via Thom isomorphism (see Section 1.3). On the other hand, let us consider the following differentiable structure on  $\mathcal{L}$ , and denote the resulting differentiable space by  $\mathcal{L}_{\text{reg}}$ :

$$\mathcal{P}(\mathcal{L}_{\text{reg}}) := \{(U, \varphi) \in \mathcal{P}(\mathcal{L}) \mid e \circ \varphi : U \rightarrow M \text{ is a submersion}\}.$$

Then, as in Section 2.3, one can define the fiber product

$$H_*^{\text{dR}}(\mathcal{L}_{\text{reg}})^{\otimes 2} \rightarrow H_{*-d}^{\text{dR}}(\mathcal{L}_{\text{reg}} \times_e \mathcal{L}_{\text{reg}}).$$

We have isomorphisms  $H_*(\mathcal{L}) \cong H_*^{\text{dR}}(\mathcal{L}) \cong H_*^{\text{dR}}(\mathcal{L}_{\text{reg}})$ ; the first isomorphism is by Theorem 4.1, and the second isomorphism is obtained by similar arguments as the proof of Lemma 5.8. We also have isomorphisms

$$H_*(\mathcal{L} \times_e \mathcal{L}) \cong H_*^{\text{dR}}(\mathcal{L} \times_e \mathcal{L}) \cong H_*^{\text{dR}}(\mathcal{L}_{\text{reg}} \times_e \mathcal{L}_{\text{reg}})$$

by similar arguments. Let us state the key technical result:

**Lemma 6.10.** *The following diagram commutes:*

$$\begin{array}{ccc} H_*^{\text{dR}}(\mathcal{L}_{\text{reg}})^{\otimes 2} & \longrightarrow & H_{*-d}^{\text{dR}}(\mathcal{L}_{\text{reg}} \times_e \mathcal{L}_{\text{reg}}) \\ \cong \downarrow & & \downarrow \cong \\ H_*(\mathcal{L})^{\otimes 2} & \longrightarrow & H_{*-d}(\mathcal{L} \times_e \mathcal{L}), \end{array}$$

where horizontal maps are fiber products.

It is easy to deduce Proposition 6.9 from Lemma 6.10, and details are left to the reader. The rest of this section is devoted to the proof of Lemma 6.10. In the next subsection, we prove Lemma 6.11, which is a preliminary result in the finite-dimensional setting.

**6.7. A preliminary result in the finite-dimensional setting.** Let  $X$  be a  $C^\infty$ -manifold, and  $e : X \rightarrow M$  be a submersion. Then, one can define the fiber product  $H_*(X)^{\otimes 2} \rightarrow H_{*-d}(X \times_e X)$  via the Thom isomorphism for the tubular neighborhood of  $X \times_e X \subset X \times X$ .

On the other hand, let us consider the following differentiable structure on  $X$ , and denote the resulting differentiable space by  $X_{\text{reg}/M}$ :

$$\mathcal{P}(X_{\text{reg}/M}) := \{(U, \varphi) \mid \varphi \in C^\infty(U, X) \text{ and } e \circ \varphi : U \rightarrow M \text{ is a submersion}\}.$$

Then, one can define the fiber product  $H_*^{\text{dR}}(X_{\text{reg}/M})^{\otimes 2} \rightarrow H_{*-d}^{\text{dR}}(X_{\text{reg}/M} \times_e X_{\text{reg}/M})$ .

It is obvious that the identity maps on  $X$  and  $X \times_e X$  induces smooth maps  $X_{\text{reg}/M} \rightarrow X$  and  $X_{\text{reg}/M} \times_e X_{\text{reg}/M} \rightarrow X \times_e X$ . These maps induce isomorphisms  $H_*^{\text{dR}}(X_{\text{reg}/M}) \cong H_*^{\text{dR}}(X)$  and  $H_*^{\text{dR}}(X_{\text{reg}/M} \times_e X_{\text{reg}/M}) \cong H_*^{\text{dR}}(X \times_e X)$ . This fact is proved by similar arguments as Proposition 3.2, and details are omitted. Then, we obtain isomorphisms

$$\begin{aligned} H_*^{\text{dR}}(X_{\text{reg}/M}) &\cong H_*^{\text{dR}}(X) \cong H_*(X), \\ H_*^{\text{dR}}(X_{\text{reg}/M} \times_e X_{\text{reg}/M}) &\cong H_*^{\text{dR}}(X \times_e X) \cong H_*(X \times_e X). \end{aligned}$$

**Lemma 6.11.** *The following diagram commutes:*

$$(9) \quad \begin{array}{ccc} H_*^{\text{dR}}(X_{\text{reg}/M})^{\otimes 2} & \longrightarrow & H_{*-d}^{\text{dR}}(X_{\text{reg}/M} \times_e X_{\text{reg}/M}) \\ \downarrow \cong & & \downarrow \cong \\ H_*(X)^{\otimes 2} & \longrightarrow & H_{*-d}(X \times_e X). \end{array}$$

**Proof.** Via isomorphisms  $H_*^{\text{dR}}(X_{\text{reg}/M}) \cong H_c^{\dim X-*}(X) \cong H_*(X)$  and

$$H_{*-d}^{\text{dR}}(X_{\text{reg}/M} \times_e X_{\text{reg}/M}) \cong H^{2\dim X-*}(X \times_e X) \cong H_{*-d}(X \times_e X),$$

both horizontal maps in (9) are identified with the map

$$H_{\text{dR}}^*(X)^{\otimes 2} \rightarrow H_{\text{dR}}^*(X \times_e X); \quad [\omega] \otimes [\eta] \mapsto [\omega \times \eta|_{X \times_e X}],$$

hence (9) is commutative.  $\square$

**6.8. Proof of Lemma 6.10.** We use notations in Section 4.1. We abbreviate  $\mathcal{L}^{\infty, E}$  by  $\mathcal{L}^E$ , and  $\mathcal{F}_N^{\infty, E}$  by  $\mathcal{F}_N^E$ . Let  $(E_j)_{j \geq 1}$  be a strictly increasing sequence of positive real numbers, such that  $\lim_{j \rightarrow \infty} E_j = \infty$ . Let us take a sequence  $(N_j)_{j \geq 1}$  of positive integers, so that  $N_j | N_{j+1}$  and  $N_j \geq N(E_j, E_{j+1})$  for every  $j \geq 1$  (see Remark 4.5). By Lemma 4.3, there exists a continuous map  $g_j : \mathcal{F}_{N_j}^{E_j} \rightarrow \mathcal{L}^{E_{j+1}}$  such that

$$(10) \quad \begin{array}{ccc} \mathcal{L}^{E_j} & \longrightarrow & \mathcal{L}^{E_{j+1}} \\ f_{N_j} \downarrow & \nearrow g_j & \downarrow f_{N_{j+1}} \\ \mathcal{F}_{N_j}^{E_j} & \longrightarrow & \mathcal{F}_{N_{j+1}}^{E_{j+1}} \end{array}$$

commutes up to homotopy. Then,  $H_*(\mathcal{L}) \cong \varinjlim_j H_*(\mathcal{L}^{E_j}) \cong \varinjlim_j H_*(\mathcal{F}_{N_j}^{E_j})$ . In the following arguments, we abbreviate  $\mathcal{F}_{N_j}^{E_j}$  by  $\mathcal{F}^j$ , and  $f_{N_j}$  by  $f_j$ .

Let  $e_j : \mathcal{F}^j \rightarrow M$ ;  $(x_k)_{0 \leq k \leq N_j} \mapsto x_0$ . As is clear from the proof of Lemma 4.3, one may take  $g_j$  so that  $e \circ g_j = e_j$  (thus  $g_j \times g_j : \mathcal{F}^j \times_{e_j \times e_j} \mathcal{F}^j \rightarrow \mathcal{L}^{E_j} \times_e \mathcal{L}^{E_j}$  is well-defined), and the following diagram commutes up to homotopy:

$$(11) \quad \begin{array}{ccc} \mathcal{L}^{E_j} \times_e \mathcal{L}^{E_j} & \longrightarrow & \mathcal{L}^{E_{j+1}} \times_e \mathcal{L}^{E_{j+1}} \\ f_j \times f_j \downarrow & \nearrow g_j \times g_j & \downarrow f_{j+1} \times f_{j+1} \\ \mathcal{F}^j \times_{e_j \times e_j} \mathcal{F}^j & \longrightarrow & \mathcal{F}^{j+1} \times_{e_{j+1} \times e_{j+1}} \mathcal{F}^{j+1}. \end{array}$$

Then,  $H_*(\mathcal{L} \times_e \mathcal{L}) \cong \varinjlim_j H_*(\mathcal{L}^{E_j} \times_e \mathcal{L}^{E_j}) \cong \varinjlim_j H_*(\mathcal{F}^j \times_{e_j \times e_j} \mathcal{F}^j)$ .

Since  $e_j : \mathcal{F}^j \rightarrow M$  is a submersion, one can define the differentiable space  $\mathcal{F}_{\text{reg}/M}^j$  as in the previous subsection. Then,  $f_j$  maps plots of  $\mathcal{L}_{\text{reg}}^{E_j}$  to plots of  $\mathcal{F}_{\text{reg}/M}^j$ . Also, one may take  $g_j$  so that it maps plots of  $\mathcal{F}_{\text{reg}/M}^j$  to plots of  $\mathcal{L}_{\text{reg}}^{E_{j+1}}$ , and the diagrams (10) and (11)

commute up to smooth homotopy with these differentiable structures. Hence we obtain

$$\begin{aligned} H_*^{\text{dR}}(\mathcal{L}_{\text{reg}}) &\cong \varinjlim_j H_*^{\text{dR}}(\mathcal{L}_{\text{reg}}^{E_j}) \cong \varinjlim_j H_*^{\text{dR}}(\mathcal{F}_{\text{reg}/M}^j), \\ H_*^{\text{dR}}(\mathcal{L}_{\text{reg}} \times_e \mathcal{L}_{\text{reg}}) &\cong \varinjlim_j H_*^{\text{dR}}(\mathcal{L}_{\text{reg}}^{E_j} \times_e \mathcal{L}_{\text{reg}}^{E_j}) \cong \varinjlim_j H_*^{\text{dR}}(\mathcal{F}_{\text{reg}/M}^j \times_{e_j} \mathcal{F}_{\text{reg}/M}^j). \end{aligned}$$

For every  $j \geq 1$ , let us consider the following diagram:  
(12)

$$\begin{array}{ccccc} H_*^{\text{dR}}(\mathcal{L}_{\text{reg}}^{E_j})^{\otimes 2} & \xrightarrow{\quad\quad\quad} & & & H_{*-d}^{\text{dR}}(\mathcal{L}_{\text{reg}}^{E_j} \times_e \mathcal{L}_{\text{reg}}^{E_j}) \\ \downarrow & \searrow \star & & \swarrow \star & \downarrow \\ & H_*^{\text{dR}}(\mathcal{F}_{\text{reg}/M}^j)^{\otimes 2} & \longrightarrow & H_{*-d}^{\text{dR}}(\mathcal{F}_{\text{reg}/M}^j \times_{e_j} \mathcal{F}_{\text{reg}/M}^j) & \\ & \downarrow & & \downarrow & \\ & H_*(\mathcal{F}^j)^{\otimes 2} & \longrightarrow & H_{*-d}(\mathcal{F}^j \times_{e_j} \mathcal{F}^j) & \\ \uparrow \star & & & & \nwarrow \star \\ H_*(\mathcal{L}^{E_j})^{\otimes 2} & \xrightarrow{\quad\quad\quad} & & & H_{*-d}(\mathcal{L}^{E_j} \times_e \mathcal{L}^{E_j}). \end{array}$$

The center square in (12) is commutative by Lemma 6.11. The commutativity of the other four squares in (12) is easy to check from definitions.

Taking direct limits as  $j \rightarrow \infty$ , all maps in (12) pass to the limit. Moreover, the limits of the maps with  $\star$  are isomorphisms. Therefore, the limit of the big square in (12) is commutative, and this completes the proof of Lemma 6.10.

## 7. GRAPHS

The rest of this paper is devoted to the proof of Theorem 6.7. In this section, we fix definitions and terminologies about (ribbon) graphs, partially following [22] and [30].

**7.1. Graphs.** As a definition of a graph, we adopt the one in [30]. A *graph* is a quadruple  $\Gamma = (V_\Gamma, F_\Gamma, \lambda_\Gamma, \iota_\Gamma)$  where  $F_\Gamma$  and  $V_\Gamma$  are finite sets,  $\lambda_\Gamma$  is a map  $F_\Gamma \rightarrow V_\Gamma$ ,  $\iota_\Gamma$  is a map  $F_\Gamma \rightarrow F_\Gamma$  such that  $\iota_\Gamma^2 = \text{id}_{F_\Gamma}$ . When there is no risk of confusion, the subscript  $\Gamma$  will be dropped.

- The elements of  $V$  are called the *vertices* of  $\Gamma$ .
- The elements of  $F$  are called the *flags* of  $\Gamma$ .
- For any  $v \in V$ , the elements of  $\lambda^{-1}(v)$  are called the flags at  $v$ .  $\#\lambda^{-1}(v)$  is called the *valence* of  $v$ , and denoted by  $\text{val}(v)$ .
- An *edge* of  $\Gamma$  is a set  $\{f, \iota(f)\}$  such that  $f \in F$  and  $f \neq \iota(f)$ . The set of all edges of  $\Gamma$  is denoted by  $E_\Gamma$ . We say that the edge  $\{f, \iota(f)\}$  connects vertices  $\lambda(f)$  and  $\lambda(\iota(f))$ .
- A *tail* of  $\Gamma$  is  $f \in F$  such that  $f = \iota(f)$ . The set of all tails of  $\Gamma$  is denoted by  $T_\Gamma$ .

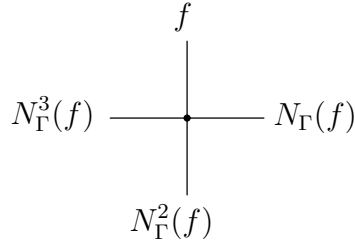
An isomorphism of graphs consists of bijections between the sets of vertices and flags, compatible with maps  $\lambda$  and  $\iota$ .

To any graph  $\Gamma$ , one can associate a 1-dimensional CW complex  $|\Gamma|$  in the obvious way (see [30], pp.1502).  $\Gamma$  is called *connected* if  $|\Gamma|$  is connected as a topological space.

For any  $E \subset E_\Gamma$ , let  $\tilde{E} := \{f \in F_\Gamma \mid \{f, \iota_\Gamma(f)\} \in E\}$ . Let us denote a graph  $(V_\Gamma, \tilde{E}, \lambda_\Gamma|_{\tilde{E}}, \iota_\Gamma|_{\tilde{E}})$  by  $\Gamma_E$ .  $E$  is called *acyclic* if  $H_1(|\Gamma_E|) = 0$ .

In this paper, a *tree* is a connected graph  $\Gamma$  which has no tails, and  $\sharp V_\Gamma - \sharp E_\Gamma = 1$ . It is easy to see that  $|\Gamma|$  is contractible. In particular, for any vertices  $v, w \in V_\Gamma$ , there exists at most one edge which connects  $v$  and  $w$ . When  $v = w$ , there is no such edge.

**7.2. Ribbon graphs.** A *ribbon graph* is a graph with a cyclic order on the set of flags at  $v$  for each vertex  $v$ . Formally, a ribbon graph is a quintuple  $\Gamma = (V_\Gamma, F_\Gamma, \lambda_\Gamma, \iota_\Gamma, N_\Gamma)$ , where  $(V_\Gamma, F_\Gamma, \lambda_\Gamma, \iota_\Gamma)$  is a graph and  $N_\Gamma : F_\Gamma \rightarrow F_\Gamma$  is a bijection, such that  $\lambda_\Gamma \circ N_\Gamma = \lambda_\Gamma$ , and  $N_\Gamma$  acts transitively on  $\lambda_\Gamma^{-1}(v)$  for every  $v \in V_\Gamma$ . An isomorphism of ribbon graphs consists of bijections between the sets of vertices and flags, compatible with maps  $\lambda$ ,  $\iota$  and  $N$ . In all figures in this paper, the map  $N_\Gamma$  is a *clockwise* rotation of flags.



Orbits of the bijection  $N_\Gamma \circ \iota_\Gamma : F_\Gamma \rightarrow F_\Gamma$  are called *cycles* of  $\Gamma$ . For any cycle  $c$ , we define a cyclic order on  $c$  by  $f < N_\Gamma \circ \iota_\Gamma(f) \ (\forall f \in c)$ . The set of all cycles of  $\Gamma$  is denoted by  $c_\Gamma$ . When  $\Gamma$  is connected, The *genus* of  $\Gamma$ , denoted by  $g(\Gamma)$ , is defined by the formula  $2 - 2g(\Gamma) := \sharp V_\Gamma - \sharp E_\Gamma + \sharp c_\Gamma$ .

**7.3. Removing flags.** Let  $\Gamma$  be a ribbon graph. For any  $F \subset F_\Gamma$ , we define a ribbon graph  $\Gamma \setminus F$  by removing elements of  $F$ . Here we present a formal definition.

First, we set  $\Gamma \setminus \emptyset := \Gamma$ . Next, we consider the case that  $F$  consists of a single flag  $f$ . We define a ribbon graph  $\Gamma \setminus \{f\}$  by  $V_{\Gamma \setminus \{f\}} := V_\Gamma$ ,  $F_{\Gamma \setminus \{f\}} := F_\Gamma \setminus \{f\}$ ,  $\lambda_{\Gamma \setminus \{f\}} := \lambda_\Gamma|_{F_\Gamma \setminus \{f\}}$ , and

$$\iota_{\Gamma \setminus \{f\}}(f') := \begin{cases} f' & (\iota_\Gamma(f') = f) \\ \iota_\Gamma(f') & (\text{otherwise}), \end{cases} \quad N_{\Gamma \setminus \{f\}}(f') := \begin{cases} N_\Gamma(f) & (N_\Gamma(f') = f) \\ N_\Gamma(f') & (\text{otherwise}). \end{cases}$$

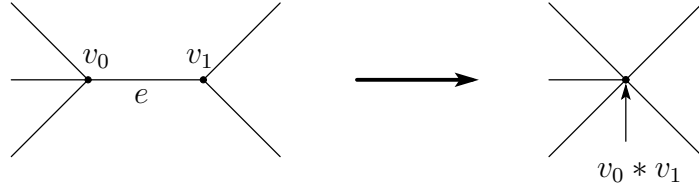
Finally, in the general case, we define so that there holds  $\Gamma \setminus (F \cup \{f\}) = (\Gamma \setminus F) \setminus \{f\}$  for any  $F \subset F_\Gamma$  and  $f \in F_\Gamma \setminus F$ .

**7.4. Contracting edges.** Let  $\Gamma$  be a ribbon graph. For any  $E \subset E_\Gamma$  which is acyclic, we define a ribbon graph  $\Gamma/E$  by contracting elements of  $E$ . Here we present a formal definition.

First, we set  $\Gamma/\emptyset := \Gamma$ . Next, we consider the case that  $E$  consists of a single edge  $e$ . Since  $E$  is acyclic,  $e$  connects different vertices. Namely, let  $e = \{f_0, f_1\}$ ,  $v_0 = \lambda_\Gamma(f_0)$ ,  $v_1 = \lambda_\Gamma(f_1)$ , then  $v_0 \neq v_1$ . We consider a new vertex  $v_0 * v_1$ , and define  $V_{\Gamma/\{e\}} := (V_\Gamma \setminus \{v_0, v_1\}) \sqcup \{v_0 * v_1\}$ . We define  $F_{\Gamma/\{e\}} := F_\Gamma \setminus e$ ,  $\iota_{\Gamma/\{e\}} := \iota_\Gamma|_{F_\Gamma \setminus e}$ , and

$$\lambda_{\Gamma/\{e\}}(f) := \begin{cases} v_0 * v_1 & (\lambda_\Gamma(f) \in \{v_0, v_1\}) \\ \lambda_\Gamma(f) & (\text{otherwise}), \end{cases}$$

$$N_{\Gamma/\{e\}}(f) := \begin{cases} N_\Gamma(f_1) & (N_\Gamma(f) = f_0, N_\Gamma(f_1) \neq f_1 \text{ or } N_\Gamma(f) = f_1, N_\Gamma(f_0) = f_0) \\ N_\Gamma(f_0) & (N_\Gamma(f) = f_1, N_\Gamma(f_0) \neq f_0 \text{ or } N_\Gamma(f) = f_0, N_\Gamma(f_1) = f_1) \\ N_\Gamma(f) & (\text{otherwise}). \end{cases}$$



contracting an edge  $e$ .

Finally, in the general case we define so that there holds  $\Gamma/(E \cup \{e\}) = (\Gamma/E)/\{e\}$  for any  $E \subset E_\Gamma$  and  $e \in E_\Gamma \setminus E$  such that  $E \cup \{e\}$  is acyclic. The next lemma is easy to check and its proof is omitted.

**Lemma 7.1.** *Let  $\Gamma$  be a ribbon graph,  $E \subset E_\Gamma$ ,  $F \subset F_\Gamma$ . Suppose that  $E$  is acyclic and  $\tilde{E} \cap F = \emptyset$ . Then,  $(\Gamma/E) \setminus F = (\Gamma \setminus F)/E$ .*

## 8. DECORATED CACTI AND FRAMING

In this section we introduce a notion of a *decorated cactus*, which is a ribbon graph with additional data. We also define the framed version (*framed decorated cactus*), and define compositions of framed decorated cacti. In the next section, we show that the collection of framed decorated cacti has a natural structure of a  $\mathbb{Z}_{\geq 0}$ -colored operad.

### 8.1. Decorated cacti.

**Definition 8.1.** Let  $\Gamma$  be a ribbon graph,  $c_0, c_1, \dots, c_r$  be cycles of  $\Gamma$ , and  $t \in T_\Gamma$ . A tuple  $(\Gamma, c_0, c_1, \dots, c_r, t)$  is called a *decorated cactus*, if the following conditions hold:

- (i):  $\Gamma$  is connected, and  $g(\Gamma) = 0$ .
- (ii):  $F_\Gamma$  is a disjoint union of  $c_0, c_1, \dots, c_r$ .
- (iii): For any  $f \in F_\Gamma$ , there holds  $\sharp(\{f, \iota_\Gamma(f)\} \cap c_0) = 1$ . In particular,  $T_\Gamma \subset c_0$ .

An isomorphism of decorated cacti  $(\Gamma, c_0, \dots, c_r, t) \rightarrow (\Gamma', c'_0, \dots, c'_r, t')$  is an isomorphism  $\Gamma \rightarrow \Gamma'$  of ribbon graphs, which maps  $t$  to  $t'$ , and  $c_i$  to  $c'_i$  for every  $i = 0, \dots, r$ .

For any  $k, l_1, \dots, l_r \in \mathbb{Z}_{\geq 0}$ ,  $\Lambda(k : l_1, \dots, l_r)$  denotes the set of isomorphism classes of decorated cacti  $(\Gamma, c_0, c_1, \dots, c_r, t)$  such that  $\sharp T_\Gamma = k + 1$ ,  $\sharp c_i = l_i + 1$  ( $i = 1, \dots, r$ ).

Given a decorated cactus  $(\Gamma, c_0, \dots, c_r, t)$ , we define a graph  $T$  by  $V_T := \{c_1, \dots, c_r\} \cup V_\Gamma$ ,  $E_T := c_1 \cup \dots \cup c_r$ , and each  $f \in c_j \subset E_T$  connects  $c_j$  and  $\lambda_\Gamma(f)$ . (Here  $c_j$  and  $\lambda_\Gamma(f)$  are considered as vertices of  $T$ .)

**Lemma 8.2.**  *$T$  is a tree.*

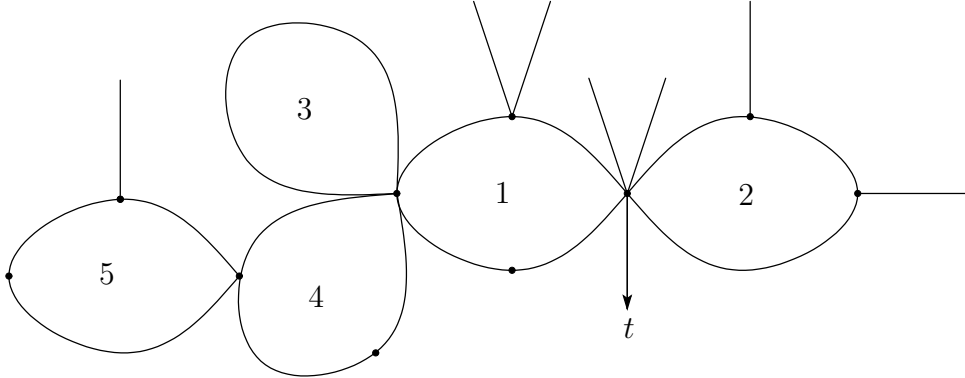
**Proof.** It is easy to see that  $T$  is connected. Therefore, it is enough to show that  $\sharp V_T - \sharp E_T = 1$ . It is obvious that  $\sharp V_T = \sharp V_\Gamma + r$ . On the other hand, Definition 8.1 (ii), (iii) imply  $\sharp E_T = \sharp c_1 + \dots + \sharp c_r = \sharp E_\Gamma$ . Therefore, we obtain  $\sharp V_T - \sharp E_T = 1 - 2g(\Gamma) = 1$ .  $\square$

We call  $T$  the *dual tree* of the decorated cactus  $(\Gamma, c_0, \dots, c_r, t)$ .

**Corollary 8.3.** *For any  $i = 1, \dots, r$  and  $f, f' \in c_i$ , if  $\lambda(f) = \lambda(f')$ , then  $f = f'$ .*

**Proof.** Let  $v := \lambda(f) = \lambda(f')$ . Then, both  $f$  and  $f'$  connect  $v$  and  $c_i$  in  $T$ . (Here  $v$  and  $c_i$  are considered as vertices of  $T$ .) Since  $T$  is a tree, we obtain  $f = f'$ .  $\square$

The figure below shows an example of a decorated cactus. This is an element in  $\Lambda(7 : 3, 2, 0, 2, 2)$ , and let us denote it by  $(\Gamma, c_0, \dots, c_5, t)$ . For each  $i = 1, \dots, 5$ ,  $c_i$  is a boundary of the region  $i$ . Notice that there are  $l_i + 1$  vertices on  $c_i$  for each  $i = 1, \dots, 5$ . The elements of  $T_\Gamma$  are depicted as the eight segments. In particular,  $t \in T_\Gamma$  is depicted as the arrow.



The configuration of cycles  $c_1, \dots, c_5$  is tree-like (this holds in general by Lemma 8.2), which is often called a “cactus” (see [29], [20]). Additionally, there are  $l_i + 1$  vertices on each cycle  $c_i$ , and  $k + 1$  tails (hence the name “decorated” cactus). The tail  $t$  corresponds to the “global zero” of a cactus (see [20] Section 2.2.6).

**Remark 8.4.** In most papers, a cactus is a tree-like configuration of *parametrized* circles. On the other hand, cycles  $c_1, \dots, c_r$  of our decorated cactus are *not* parametrized. In the next subsection, we introduce a notion of a *framing* on a decorated cactus, which corresponds to a parametrization of a circle.

The next lemma would be clear from geometric intuitions, however we give a detailed proof.

**Lemma 8.5.** *Let  $i = 1, \dots, r$  and  $f_0 \in c_i$ . For each  $j = 1, \dots, l_i$ , let  $f_j := (N \circ \iota)^j(f_0)$ . Then,  $\iota(f_0) > \iota(f_1) > \dots > \iota(f_{l_i}) > \iota(f_0)$  with respect to the cyclic order on  $c_0$ .*

**Proof.** It is enough to prove the following claim:

Suppose that  $f \in c_i$  and  $k \geq 1$  satisfy  $(\iota \circ N)^k(f) \in c_i$  and  $(\iota \circ N)^l(f) \notin c_i$  for any  $l = 1, \dots, k-1$ . Then,  $f = N \circ \iota \circ (\iota \circ N)^k(f)$ .

Let  $f' := N \circ \iota \circ (\iota \circ N)^k(f)$ . Since  $f, f' \in c_i$ , by Corollary 8.3, it is enough to show that  $\lambda(f) = \lambda(f')$ . For any  $l = 0, \dots, k-1$ , let  $v_l := \lambda((\iota \circ N)^l(f))$ . Then,  $v_0 = \lambda(f)$ , and

$$v_{k-1} = \lambda((\iota \circ N)^{k-1}(f)) = \lambda(N^2 \circ (\iota \circ N)^{k-1}(f)) = \lambda(N \circ \iota \circ (\iota \circ N)^k(f)) = \lambda(f').$$

Therefore, we have to show  $v_0 = v_{k-1}$ . Since this is obvious for  $k = 1$ , we may assume that  $k > 1$ .

Let  $T$  be the dual tree of  $\Gamma$ . For each  $l = 1, \dots, k-1$ , let us take  $j$  so that  $(\iota \circ N)^l(f) \in c_j$ . Then,  $v_l = \lambda((\iota \circ N)^l(f)) \in \lambda(c_j)$ , and  $v_{l-1} = \lambda(N \circ \iota \circ (\iota \circ N)^{l-1}(f)) \in \lambda(c_j)$ . Therefore, if  $j \neq 0$ , then  $v_{l-1} - c_j - v_l$  is a path on  $T$ . If  $j = 0$ , then  $(\iota \circ N)^l(f)$  is a tail, since  $\iota(\iota \circ N)^l(f) = (N \circ \iota)^l(\iota f) \in c_0$ . Therefore,  $v_{l-1} = v_l$ . In conclusion, there exists a path from  $v_0$  to  $v_{k-1}$  on  $T$ , which is disjoint from  $c_i$ .

On the other hand, since  $v_0 = \lambda(f) \in \lambda(c_i)$  and  $v_{k-1} = \lambda(f') \in \lambda(c_i)$ ,  $v_0 - c_i - v_{k-1}$  is a path on  $T$ . Since  $T$  is a tree, we conclude that  $v_0 = v_{k-1}$ .  $\square$

**8.2. Framing on decorated cacti.** A *framing* on a decorated cactus  $(\Gamma, c_0, \dots, c_r, t)$  is a map  $\text{fr} : \{c_1, \dots, c_r\} \rightarrow F_\Gamma$ , such that  $\text{fr}(c_i) \in c_i$  for every  $i = 1, \dots, r$ . A tuple  $(\Gamma, c_0, \dots, c_r, t, \text{fr})$  is called a *framed decorated cactus*. An isomorphism of framed decorated cacti is an isomorphism of decorated cacti which preserves framing. For any  $k, l_1, \dots, l_r \in \mathbb{Z}_{\geq 0}$ ,  $f\Lambda(k : l_1, \dots, l_r)$  denotes the set of isomorphism classes of framed decorated cacti  $(x, \text{fr})$  such that  $[x] \in \Lambda(k : l_1, \dots, l_r)$ .

Let  $(\Gamma, c_0, \dots, c_r, t)$  be a decorated cactus, and  $T$  be its dual tree. There exists a unique framing  $\text{fr} : \{c_1, \dots, c_r\} \rightarrow F_\Gamma$  such that, for every  $i = 1, \dots, r$ ,  $\text{fr}(c_i)$  is on the shortest path of  $T$  connecting  $c_i$  and  $\lambda(t)$  (here  $\text{fr}(c_i)$  is considered as an edge of  $T$ , and  $c_i, \lambda(t)$  are considered as vertices of  $T$ ). We call it the *canonical framing*. By considering canonical framings, we define an injective map  $\Lambda(k : l_1, \dots, l_r) \rightarrow f\Lambda(k : l_1, \dots, l_r)$ . In the rest of this paper, we consider  $\Lambda(k : l_1, \dots, l_r)$  as a subset of  $f\Lambda(k : l_1, \dots, l_r)$ .

When we draw a figure of a framed decorated cactus  $(\Gamma, c_0, \dots, c_r, t, \text{fr})$ , we draw an arrow pointing at a vertex  $\lambda(\text{fr}(c_i))$  for each  $i = 1, \dots, r$ .

**8.3. Actions of the symmetric groups.** For any  $\sigma \in \mathbb{S}_r$  and  $x = (\Gamma, c_0, c_1, \dots, c_r, t, \text{fr}) \in f\Lambda(k : l_1, \dots, l_r)$ , we set  $x^\sigma := (\Gamma, c_0, c_{\sigma(1)}, \dots, c_{\sigma(r)}, t, \text{fr}^\sigma)$ , where  $\text{fr}^\sigma(i) := \text{fr}(\sigma(i))$  for every  $i = 1, \dots, r$ . Then, a map

$$f\Lambda(k : l_1, \dots, l_r) \rightarrow f\Lambda(k : l_{\sigma(1)}, \dots, l_{\sigma(r)}); \quad x \mapsto x^\sigma$$

is a bijection, and there holds  $(x^\sigma)^\tau = x^{\sigma\circ\tau}$  for any  $\sigma, \tau \in \mathbb{S}_r$ . As a restriction of this map, we obtain a bijection  $\Lambda(k : l_1, \dots, l_r) \rightarrow \Lambda(k : l_{\sigma(1)}, \dots, l_{\sigma(r)})$ .

**8.4. Examples of framed decorated cacti.** Here we present examples of framed decorated cacti. For any decorated cactus  $x = (\Gamma, c_0, \dots, c_r, t)$ , we denote  $V_\Gamma, F_\Gamma$  etc. as  $V_x, F_x$  etc. Moreover,  $c_0, \dots, c_r$  are denoted as  $c_0^x, \dots, c_r^x$ , and  $t$  is denoted as  $t_x$ .



For any  $k \geq 0$ ,  $\Lambda(k : ) = f\Lambda(k : )$  consists of a unique element, which we denote by  $\zeta_k$ . More explicitly,  $\zeta_k$  is defined as follows:

- $V_{\zeta_k} := \{v_0\}$ ,  $F_{\zeta_k} := \{t_0, \dots, t_k\}$ .
- For any  $i = 0, \dots, k$ ,

$$\lambda_{\zeta_k}(t_i) := v_0, \quad N_{\zeta_k}(t_i) := \begin{cases} t_{i+1} & (0 \leq i < k) \\ t_0 & (i = k), \end{cases} \quad \iota_{\zeta_k}(t_i) := t_i.$$

- $c_0^{\zeta_k} := F_{\zeta_k}$ ,  $t_{\zeta_k} := t_0$ .



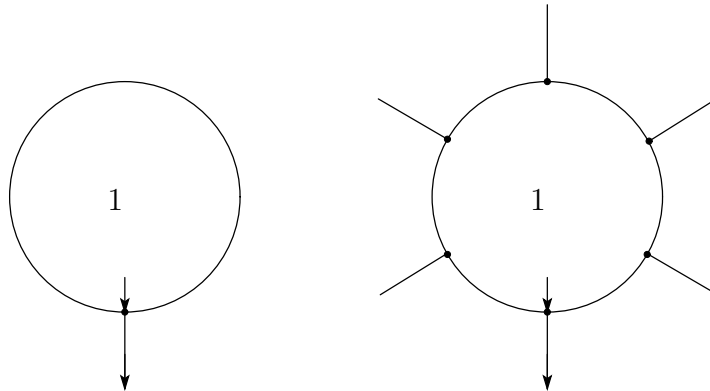
$\zeta_0$  (left) and  $\zeta_5$  (right)

For any  $k \geq 0$ , there exists a unique element  $\varepsilon_k \in \Lambda(k : k)$ , such that  $\lambda_{\varepsilon_k}|_{T_{\varepsilon_k}} : T_{\varepsilon_k} \rightarrow V_{\varepsilon_k}$  is a bijection. More explicitly,  $\varepsilon_k$  is defined as follows:

- $V_{\varepsilon_k} := \{v_0, \dots, v_k\}$ ,  $F_{\varepsilon_k} := \{f_0^-, \dots, f_k^-, f_0^+, \dots, f_k^+, t_0, \dots, t_k\}$ .
- For any  $i = 0, \dots, k$ , we define

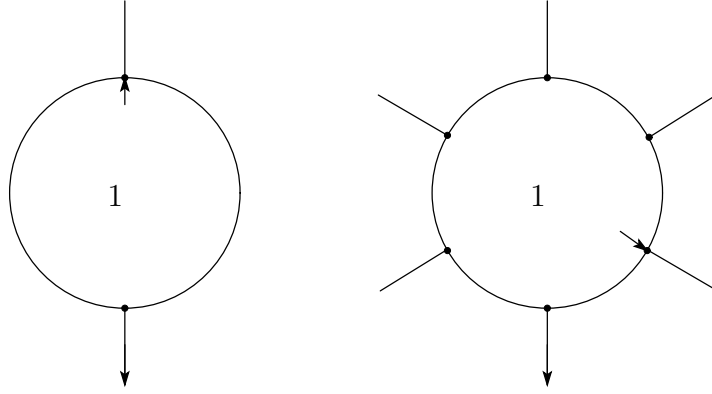
$$\begin{aligned} \lambda_{\varepsilon_k}(f_i^-) &= \lambda_{\varepsilon_k}(f_i^+) = \lambda_{\varepsilon_k}(t_i) = v_i, \\ N_{\varepsilon_k}(f_i^-) &= t_i, \quad N_{\varepsilon_k}(f_i^+) = f_i^-, \quad N_{\varepsilon_k}(t_i) = f_i^+, \\ \iota_{\varepsilon_k}(t_i) &= t_i, \quad \iota_{\varepsilon_k}(f_i^+) = \begin{cases} f_{i+1}^- & (0 \leq i < k) \\ f_0^- & (i = k), \end{cases} \quad \iota_{\varepsilon_k}(f_i^-) = \begin{cases} f_{i-1}^+ & (0 < i \leq k) \\ f_k^+ & (i = 0). \end{cases} \end{aligned}$$

- $c_0^{\varepsilon_k} = \{t_0, \dots, t_k, f_0^+, \dots, f_k^+\}$ ,  $c_1^{\varepsilon_k} = \{f_0^-, \dots, f_k^-\}$ .
- $t_{\varepsilon_k} = t_0$ .



$\varepsilon_0$  (left) and  $\varepsilon_5$  (right)

For any  $k \geq 1$ , let us consider a noncanonical framing  $\text{fr}$  on  $\varepsilon_k$  so that  $\text{fr}(c_1) = N \circ \iota \circ N^2(t)$ . We obtain an element in  $f\Lambda(k : k)$ , which we denote by  $\tau_k$ .

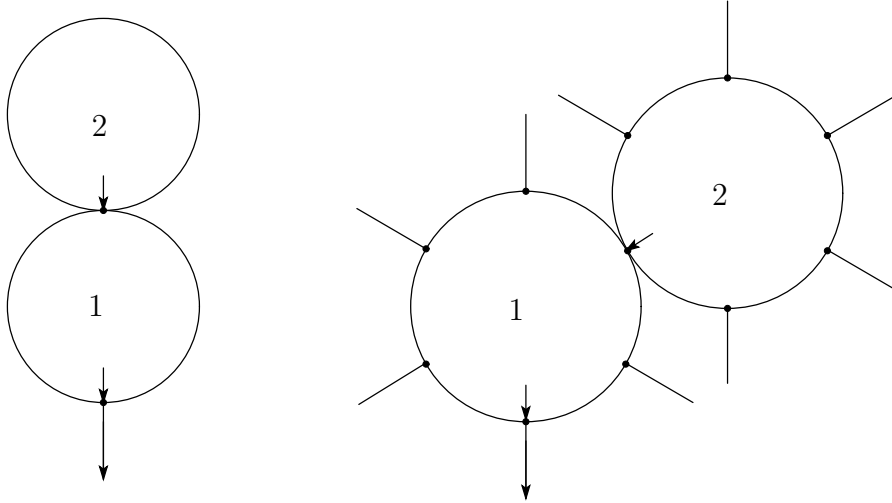


$\tau_1$  (left) and  $\tau_5$  (right)

Finally, for any  $k \geq i \geq 1$  and  $l \geq 0$ , we define  $\beta_{k,i,l} \in \Lambda(k+l-1 : k, l)$ . In general, for any  $x \in \Lambda(m : k, l)$  where  $k, l, m \geq 0$ ,  $\lambda_x(c_1^x) \cap \lambda_x(c_2^x)$  consists of a unique vertex (consider its dual tree), which we denote by  $v_x$ . Let us consider the following condition:

( $*$ ):  $v_x \notin \lambda_x(T_x)$ , and  $\lambda_x|_{T_x} : T_x \rightarrow V_x \setminus \{v_x\}$  is a bijection.

For any  $k, l \geq 0$ ,  $\Lambda(k+l-1 : k, l)$  has exactly  $k+l$  elements which satisfy ( $*$ ), depending on choices of  $t_x \in T_x$ . In particular, for every  $i = 1, \dots, k$ , there exists a unique element  $x \in \Lambda(k+l-1 : k, l)$  which satisfies ( $*$ ),  $\lambda_x(t_x) \in \lambda_x(c_1^x)$  and  $v_x = \lambda_x((N_x \circ \iota_x)^{2i}(t_x))$ . We denote this element as  $\beta_{k,i,l}$ .



$\beta_{1,1,0}$  (left) and  $\beta_{5,4,5}$  (right)

**8.5. Compositions of framed decorated cacti.** For any  $k, l_1, \dots, l_r, m_1, \dots, m_s \geq 0$  and  $i = 1, \dots, r$ , we define a composition map

$$\circ_i : f\Lambda(k : l_1, \dots, l_r) \times f\Lambda(l_i : m_1, \dots, m_s) \rightarrow f\Lambda(k : l_1, \dots, l_{i-1}, m_1, \dots, m_s, l_{i+1}, \dots, l_r).$$

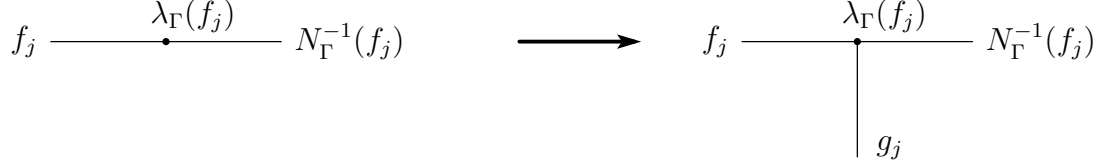
Namely, given

$$x = (\Gamma, c_0, \dots, c_r, t, \text{fr}) \in f\Lambda(k : l_1, \dots, l_r), \quad y = (\Gamma', c'_0, \dots, c'_s, t', \text{fr}') \in f\Lambda(l_i : m_1, \dots, m_s),$$

we define  $x \circ_i y \in f\Lambda(k : l_1, \dots, l_{i-1}, m_1, \dots, m_s, l_{i+1}, \dots, l_r)$ .

We first define a ribbon graph  $\Gamma *_i \Gamma'$  as follows:

- $V_{\Gamma *_i \Gamma'} := V_\Gamma \sqcup V_{\Gamma'}$ .
- For every  $j = 0, \dots, l_i$ , let  $f_j := (N_\Gamma \circ \iota_\Gamma)^j(\text{fr}(c_i)) \in c_i$ , and insert a new flag  $g_j$  at  $\lambda_\Gamma(f_j)$  as in the figure below.



Precisely, we define

$$F_{\Gamma *_i \Gamma'} := F_\Gamma \sqcup F_{\Gamma'} \sqcup \{g_0, \dots, g_{l_i}\},$$

$$\lambda_{\Gamma *_i \Gamma'}(f) := \begin{cases} \lambda_\Gamma(f) & (f \in F_\Gamma) \\ \lambda_{\Gamma'}(f) & (f \in F_{\Gamma'}) \\ \lambda_\Gamma(f_j) & (f = g_j), \end{cases}$$

$$N_{\Gamma *_i \Gamma'}(f) := \begin{cases} f_j & (f = g_j) \\ g_j & (f = N_\Gamma^{-1}(f_j)) \\ N_\Gamma(f) & (f \in F_\Gamma \setminus \iota_\Gamma(c_i)) \\ N_{\Gamma'}(f) & (f \in F_{\Gamma'}). \end{cases}$$

- For every  $j = 0, \dots, l_i$ , we define  $t'_j \in T_{\Gamma'}$  (the set of tails of  $\Gamma'$ ) so that  $t'_0 = t'$  and  $t'_0 > t'_1 > \dots > t'_{l_i} > t'_0$  with respect to the cyclic order on  $c'_0$ . We “glue”  $t'_j$  and  $g_j$  for every  $j = 0, \dots, l_i$ . Precisely, we define

$$\iota_{\Gamma *_i \Gamma'}(f) := \begin{cases} g_j & (f = t'_j) \\ t'_j & (f = g_j) \\ \iota_\Gamma(f) & (f \in F_\Gamma) \\ \iota_{\Gamma'}(f) & (f \in F_{\Gamma'} \setminus T_{\Gamma'}). \end{cases}$$

This completes the definition of  $\Gamma *_i \Gamma'$ .

For every  $j = 0, \dots, l_i$ , we define  $e_j \in E(\Gamma *_i \Gamma')$  by  $e_j := \{g_j, t'_j\}$ . We define a new ribbon graph  $\Gamma \circ_i \Gamma'$  by removing flags in  $c_i \cup \iota_\Gamma(c_i)$ , and contracting edges  $e_0, \dots, e_{l_i}$ :

$$\Gamma \circ_i \Gamma' := (\Gamma *_i \Gamma' \setminus (c_i \cup \iota_\Gamma(c_i))) / \{e_0, \dots, e_{l_i}\}.$$

There is a natural bijection  $F_{\Gamma \circ_i \Gamma'} \rightarrow (F_\Gamma \sqcup F_{\Gamma'}) \setminus (c_i \cup \iota_\Gamma(c_i) \cup T_{\Gamma'})$ . Let us identify the LHS and the RHS via this bijection.

- Lemma 8.6.** (i): For any  $j \in \{1, \dots, r\} \setminus \{i\}$ ,  $c_j$  is a cycle of  $\Gamma \circ_i \Gamma'$ .  
(ii): For any  $j \in \{1, \dots, s\}$ ,  $c'_j$  is a cycle of  $\Gamma \circ_i \Gamma'$ .  
(iii):  $c''_0 := (c_0 \setminus \iota_\Gamma(c_i)) \cup (c'_0 \setminus T_{\Gamma'})$  is a cycle of  $\Gamma \circ_i \Gamma'$ .

**Proof.** (i) holds since  $N_\Gamma \circ \iota_\Gamma(f) = N_{\Gamma \circ_i \Gamma'} \circ \iota_{\Gamma \circ_i \Gamma'}(f)$  for any  $f \in c_j$ . (ii) holds since  $N_{\Gamma'} \circ \iota_{\Gamma'}(f) = N_{\Gamma \circ_i \Gamma'} \circ \iota_{\Gamma \circ_i \Gamma'}(f)$  for any  $f \in c'_j$ . (iii) is a consequence of Lemma 8.5.  $\square$

By Lemma 8.6, we obtain  $c_{\Gamma \circ_i \Gamma'} = \{c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_r, c'_1, \dots, c'_s, c''_0\}$ . In particular,  $\sharp c_{\Gamma \circ_i \Gamma'} = r + s = \sharp c_\Gamma + \sharp c_{\Gamma'} - 2$ . On the other hand, there holds

$$\sharp V_{\Gamma \circ_i \Gamma'} = \sharp V_\Gamma + \sharp V_{\Gamma'} - (l_i + 1), \quad \sharp E_{\Gamma \circ_i \Gamma'} = \sharp E_\Gamma + \sharp E_{\Gamma'} - (l_i + 1).$$

Then, we obtain  $g(\Gamma \circ_i \Gamma') = g(\Gamma) + g(\Gamma') = 0$ . Moreover,  $\sharp(\{f, \iota_{\Gamma \circ_i \Gamma'}(f)\} \cap c''_0) = 1$  for any  $f \in F_{\Gamma \circ_i \Gamma'}$ . Also,  $T_{\Gamma \circ_i \Gamma'} = T_\Gamma$ . In conclusion,  $(\Gamma \circ_i \Gamma', c''_0, c_1, \dots, c_{i-1}, c'_1, \dots, c'_s, c_{i+1}, \dots, c_r, t)$  is a decorated cactus.

Finally, we define a framing  $\text{fr}'' : c_{\Gamma \circ_i \Gamma'} \rightarrow F_{\Gamma \circ_i \Gamma'}$  by

$$\text{fr}''(c'_j) := \text{fr}'(c'_j) \ (j = 1, \dots, s), \quad \text{fr}''(c_j) := \text{fr}(c_j) \ (j \in \{1, \dots, r\} \setminus \{i\}).$$

Now we define  $x \circ_i y$  to be the resulting framed decorated cactus. This completes the definition of compositions of framed decorated cacti.

**Lemma 8.7.** (i): For any  $x \in f\Lambda(k : l_1, \dots, l_r)$ , there holds  $\varepsilon_k \circ_1 x = x \circ_1 \varepsilon_{l_1} = \dots = x \circ_r \varepsilon_{l_r} = x$ .

(ii): For any  $1 \leq i < j \leq r$  and  $x \in f\Lambda(k : l_1, \dots, l_r)$ ,  $y \in f\Lambda(l_i : m_1, \dots, m_s)$ ,  $z \in f\Lambda(l_j : n_1, \dots, n_t)$ , there holds  $(x \circ_i y) \circ_{j+s-1} z = (x \circ_j z) \circ_i y$ .

(iii): For any  $1 \leq i \leq r$ ,  $1 \leq j \leq s$  and  $x \in f\Lambda(k : l_1, \dots, l_r)$ ,  $y \in f\Lambda(l_i : m_1, \dots, m_s)$ ,  $z \in f\Lambda(m_j : n_1, \dots, n_t)$ , there holds  $x \circ_i (y \circ_j z) = (x \circ_i y) \circ_{i+j-1} z$ .

(iv): If  $x \in \Lambda(k : l_1, \dots, l_r)$  and  $y \in \Lambda(l_i : m_1, \dots, m_s)$ , then

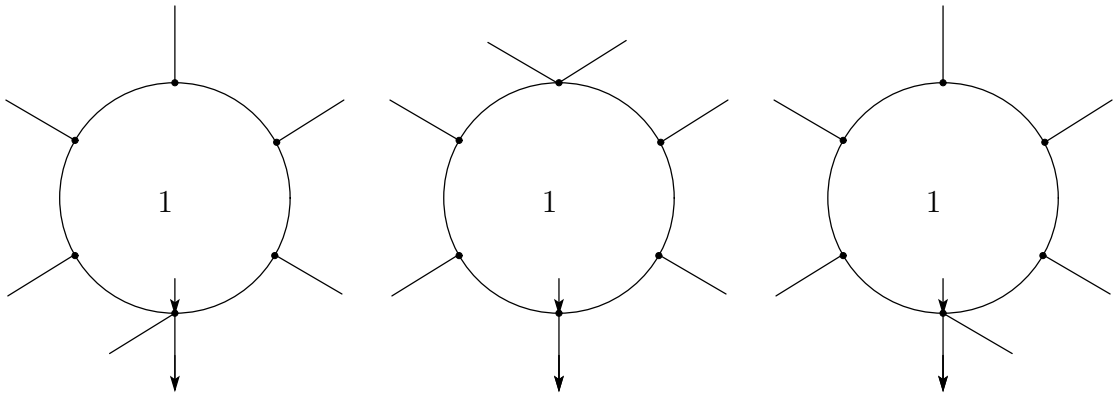
$$x \circ_i y \in \Lambda(k : l_1, \dots, l_{i-1}, m_1, \dots, m_s, l_{i+1}, \dots, l_r).$$

**Proof.** (ii), (iii) are consequences of Lemma 7.1. (i), (iv) are easy to verify, and proofs are omitted.  $\square$

**8.6. A few more examples of decorated cacti.** We end this section with a few more examples of decorated cacti.

For any  $k \geq 1$  and  $i = 0, \dots, k$ , we define  $\delta_{k,i} \in \Lambda(k : k-1)$  by

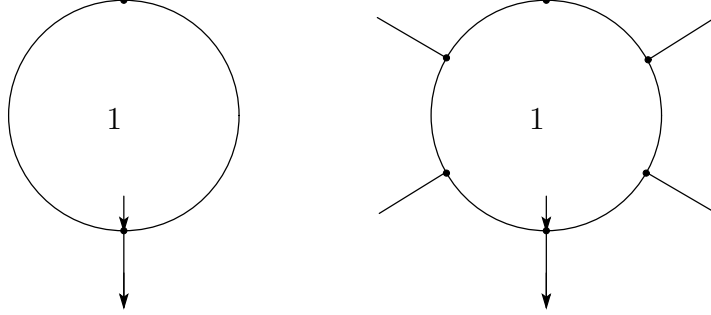
$$\delta_{k,i} := \begin{cases} \beta_{2,2,k-1} \circ_1 \zeta_2 & (i = 0) \\ \beta_{k-1,i,2} \circ_2 \zeta_2 & (1 \leq i \leq k-1) \\ \beta_{2,1,k-1} \circ_1 \zeta_2 & (i = k). \end{cases}$$



$\delta_{6,0}$  (left),  $\delta_{6,3}$  (middle) and  $\delta_{6,6}$  (right)

For any  $k \geq 0$  and  $i = 0, \dots, k$ , we define  $\sigma_{k,i} \in \Lambda(k : k+1)$  by

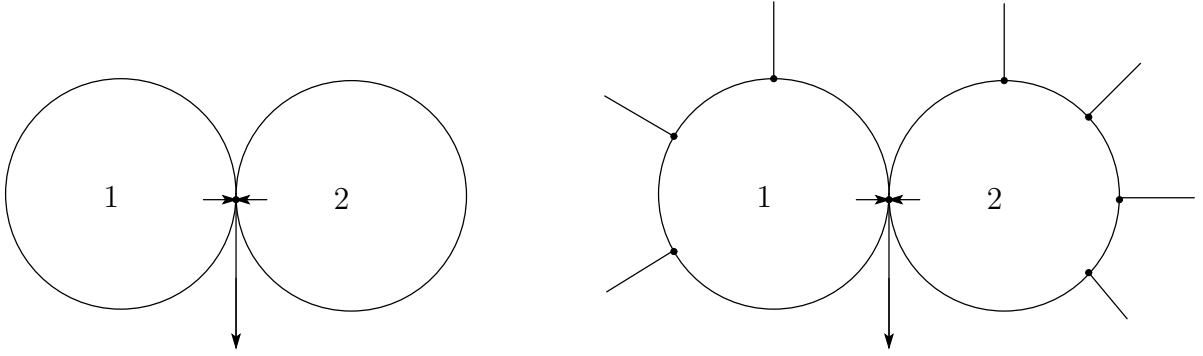
$$\sigma_{k,i} := \beta_{k+1,i+1,0} \circ_2 \zeta_0.$$



$\sigma_{0,0}$  (left) and  $\sigma_{4,2}$  (right)

For any  $k, l \geq 0$ , we define  $\alpha_{k,l} \in \Lambda(k+l : k, l)$  by

$$\alpha_{k,l} := \beta_{k+1,k+1,l} \circ_1 \delta_{k+1,k+1} = \beta_{l+1,1,k}^{(12)} \circ_2 \delta_{l+1,0}.$$



$\alpha_{0,0}$  (left) and  $\alpha_{3,4}$  (right)

## 9. COLORED OPERAD OF (FRAMED) DECORATED CACTI

Section 9.1 is devoted to some preliminaries on colored operads. In Section 9.2, we show that the collection of framed decorated cacti has a natural structure of a  $\mathbb{Z}_{\geq 0}$ -colored operad, and we give a presentation of this colored operad by generators and relations (Lemma 9.3).

**9.1. Preliminaries on colored operads.** Let  $K$  be any set, and  $\mathcal{C}$  be any symmetric monoidal category. A  $K$ -colored operad in  $\mathcal{C}$ , which we denote by  $\mathcal{P}$ , consists of the following data:

- $\mathcal{P}(k : l_1, \dots, l_r) \in \mathcal{C}$  for any  $r \geq 0$  and  $k, l_1, \dots, l_r \in K$ .
- A morphism  $e_k : 1_{\mathcal{C}} \rightarrow \mathcal{P}(k : k)$  for any  $k \in K$ , which we call a unit.
- For any  $1 \leq i \leq r$ ,  $s \geq 0$  and  $k, l_1, \dots, l_r, m_1, \dots, m_s \in K$ , a composition morphism

$$\circ_i : \mathcal{P}(k : l_1, \dots, l_r) \times \mathcal{P}(l_i : m_1, \dots, m_s) \rightarrow \mathcal{P}(k : l_1, \dots, l_{i-1}, m_1, \dots, m_s, l_{i+1}, \dots, l_r).$$

- For any  $r \geq 0$ ,  $k, l_1, \dots, l_r \in K$  and  $\sigma \in \mathbb{S}_r$ , a morphism

$$\sigma_* : \mathcal{P}(k : l_1, \dots, l_r) \rightarrow \mathcal{P}(k : l_{\sigma(1)}, \dots, l_{\sigma(r)}).$$

We require that  $(\text{id}_{\mathbb{S}_r})_* = \text{id}_{\mathcal{P}(k : l_1, \dots, l_r)}$ , and  $(\sigma \circ \tau)_* = \tau_* \circ \sigma_*$ .

We require that each  $e_k$  is a two-sided unit for composition morphisms, and there holds the associativity and equivariance properties, which are defined in the obvious ways. Notions of morphisms and suboperads for  $K$ -colored operads are also defined in the obvious ways. Colored operads in the category of dg (resp. graded) vector spaces are called dg (resp. graded)  $K$ -colored operads.

Let  $V = (V(k))_{k \in K}$  be a family of dg vector spaces. Setting

$$\text{End}(V)(k : l_1, \dots, l_r) := \text{Hom}(V(l_1) \otimes \dots \otimes V(l_r), V(k)),$$

$\text{End}(V)$  has the natural structure of a dg  $K$ -colored operad. For any dg  $K$ -colored operad  $\mathcal{P}$ , a dg  $\mathcal{P}$ -algebra consists of  $V = (V(k))_{k \in K}$  and a morphism  $\mathcal{P} \rightarrow \text{End}(V)$  of dg  $K$ -colored operads. We obtain the following chain map for any  $k, l_1, \dots, l_r \in K$ :

$$\mathcal{P}(k : l_1, \dots, l_r) \otimes V(l_1) \otimes \dots \otimes V(l_r) \rightarrow V(k); \quad x \otimes v^{l_1} \otimes \dots \otimes v^{l_r} \mapsto x \cdot (v^{l_1} \otimes \dots \otimes v^{l_r}).$$

For any dg  $\mathcal{P}$ -algebras  $V$  and  $W$ , a morphism  $\varphi : V \rightarrow W$  of dg  $\mathcal{P}$ -algebras is a family of chain maps  $\varphi = (\varphi(k) : V(k)_* \rightarrow W(k)_*)_{k \in K}$  such that

$$\varphi(k)(x \cdot (v_1 \otimes \dots \otimes v_r)) = x \cdot (\varphi(l_1)(v_1) \otimes \dots \otimes \varphi(l_r)(v_r))$$

for any  $x \in \mathcal{P}(k : l_1, \dots, l_r)$  and  $v_i \in V(l_i)$  ( $i = 1, \dots, r$ ).

**Lemma 9.1.** *Let  $S = (S(k : l_1, \dots, l_r))_{k, l_1, \dots, l_r \in K}$  be a family of sets. There exists a unique (up to isomorphism)  $K$ -colored operad  $\mathcal{F}_S$  of sets, which satisfies the following conditions:*

- For any  $k, l_1, \dots, l_r \in K$ , there exists a map

$$\varphi_{k : l_1, \dots, l_r}^S : S(k : l_1, \dots, l_r) \rightarrow \mathcal{F}_S(k : l_1, \dots, l_r).$$

- Let  $\mathcal{P}$  be any  $K$ -colored operad of sets, such that for any  $k, l_1, \dots, l_r \in K$ , a map  $\varphi_{k : l_1, \dots, l_r} : S(k : l_1, \dots, l_r) \rightarrow \mathcal{P}(k : l_1, \dots, l_r)$  is given. Then, there exists a unique morphism of  $K$ -colored operads  $\psi : \mathcal{F}_S \rightarrow \mathcal{P}$ , such that  $\varphi = \psi \circ \varphi^S$ .

We call  $\mathcal{F}_S$  the free  $K$ -colored operad generated by  $S$ .

**Proof.** Uniqueness is obvious. Existence is proved by an explicit construction using planar trees with all edges colored by  $K$ . The construction is standard (see e.g. [3]) and omitted.  $\square$

Let  $\mathcal{P}$  be a  $K$ -colored operad of sets. A binary relation on  $\mathcal{P}$  is a family  $R = (R(k : l_1, \dots, l_r))_{k, l_1, \dots, l_r \in K}$ , where each  $R(k : l_1, \dots, l_r)$  is a binary relation on  $\mathcal{P}(k : l_1, \dots, l_r)$ . If  $R(k : l_1, \dots, l_r)$  is an equivalence relation for any  $k, l_1, \dots, l_r$ , and there holds

$$x \sim_R y \implies z \circ_i x \sim_R z \circ_i y, \quad x \circ_j w \sim_R y \circ_j w, \quad x^\sigma \sim_R y^\sigma$$

for any  $x, y \in \mathcal{P}(l_i : m_1, \dots, m_s)$ ,  $z \in \mathcal{P}(k : l_1, \dots, l_r)$ ,  $w \in \mathcal{P}(m_j : n_1, \dots, n_t)$  and  $\sigma \in \mathbb{S}_s$ ,  $R$  is called an equivalence relation on  $\mathcal{P}$ . For any equivalence relation  $R$  on  $\mathcal{P}$ ,  $\mathcal{P}/\sim_R := (\mathcal{P}(k : l_1, \dots, l_r)/\sim_R)_{k, l_1, \dots, l_r \in K}$  has a natural  $K$ -colored operad structure.

Let  $S = (S(k : l_1, \dots, l_r))_{k, l_1, \dots, l_r \in K}$  be a family of sets, and  $R = (R(k : l_1, \dots, l_r))_{k, l_1, \dots, l_r \in K}$  be a binary relation on the free  $K$ -colored operad  $\mathcal{F}_S$ . Let  $\bar{R}$  denote the equivalence relation on  $\mathcal{F}_S$  generated by  $R$  (i.e. the minimal equivalence relation which contains  $R$ ). Then, we define  $\langle S \mid R \rangle := \mathcal{F}_S / \sim_{\bar{R}}$ .

**9.2. Colored operad structure on (framed) decorated cacti.** Let us denote

$$f\Lambda := (f\Lambda(k : l_1, \dots, l_r))_{k, l_1, \dots, l_r \in \mathbb{Z}_{\geq 0}}, \quad \Lambda := (\Lambda(k : l_1, \dots, l_r))_{k, l_1, \dots, l_r \in \mathbb{Z}_{\geq 0}}.$$

We can rephrase Lemma 8.7 as follows. (Compatibility between symmetric group actions and composition maps is not verified there, however this is obvious.)

**Proposition 9.2.**  *$f\Lambda$  is a  $\mathbb{Z}_{\geq 0}$ -colored operad with the composition maps and symmetric group actions defined in Section 8, and unit elements are  $\varepsilon_k$  ( $\forall k \geq 0$ ).  $\Lambda$  is its suboperad.*

The aim of this subsection is to give explicit presentations of  $\Lambda$  and  $f\Lambda$  by generators and relations. For any  $k, l_1, \dots, l_r \in \mathbb{Z}_{\geq 0}$ , let us define

$$\begin{aligned} B(k : l_1, \dots, l_r) &:= \begin{cases} \{\beta_{l_1, i, l_2} \mid 1 \leq i \leq l_1\} & (r = 2, k = l_1 + l_2 - 1) \\ \emptyset & (\text{otherwise}), \end{cases} \\ Z(k : l_1, \dots, l_r) &:= \begin{cases} \{\zeta_k\} & (r = 0) \\ \emptyset & (\text{otherwise}), \end{cases} \\ T(k : l_1, \dots, l_r) &:= \begin{cases} \{\tau_k\} & (r = 1, k = l_1 \geq 1) \\ \emptyset & (\text{otherwise}). \end{cases} \end{aligned}$$

It is easy to check the following relations:

$$(13) \quad j < i \implies \beta_{k+l-1, j, m} \circ_1 \beta_{k, i, l} = (\beta_{k+m-1, i+m-1, l} \circ_1 \beta_{k, j, m})^{(23)},$$

$$(14) \quad i \leq j \leq i + l - 1 \implies \beta_{k+l-1, j, m} \circ_1 \beta_{k, i, l} = \beta_{k, i, l+m-1} \circ_2 \beta_{l, j-i+1, m},$$

$$(15) \quad \beta_{1, 1, l} \circ_1 \zeta_1 = \varepsilon_l, \quad \beta_{k, i, 1} \circ_2 \zeta_1 = \varepsilon_k,$$

$$(16) \quad (\beta_{k, i, l} \circ_1 \zeta_k) \circ_1 \zeta_l = \zeta_{k+l-1},$$

$$(17) \quad \tau_{k+l-1} \circ_1 \beta_{k, i, l} = \begin{cases} \beta_{k, i-1, l} \circ_1 \tau_k & (i > 1) \\ (\beta_{l, l, k}^{(12)} \circ_2 \tau_l) \circ_1 \tau_k & (i = 1), \end{cases}$$

$$(18) \quad \underbrace{\tau_k \circ_1 \dots \circ_1 \tau_k}_{k+1 \text{ times}} = \varepsilon_k,$$

$$(19) \quad \tau_k \circ_1 \zeta_k = \zeta_k.$$

Let us define  $\mathbb{Z}_{\geq 0}$ -colored operads  $\mathcal{P}_B, \mathcal{P}_{BZ}, \mathcal{P}_{BZT}$  by generators and relations:  
 $\mathcal{P}_B := \langle B \mid (13), (14) \rangle, \quad \mathcal{P}_{BZ} := \langle BUZ \mid (13)-(16) \rangle, \quad \mathcal{P}_{BZT} := \langle BUZUT \mid (13)-(19) \rangle.$

There exist natural morphisms of  $\mathbb{Z}_{\geq 0}$ -colored operads  $\mathcal{P}_B \rightarrow \mathcal{P}_{BZ} \rightarrow \mathcal{P}_{BZT}$  and

$$\rho_B : \mathcal{P}_B \rightarrow \Lambda, \quad \rho_{BZ} : \mathcal{P}_{BZ} \rightarrow \Lambda, \quad \rho_{BZT} : \mathcal{P}_{BZT} \rightarrow f\Lambda.$$

**Lemma 9.3.**  $\rho_{BZ}, \rho_{BZT}$  are isomorphisms of  $\mathbb{Z}_{\geq 0}$ -colored operads.

**Proof.** The proof consists of six steps.

**Step 1.** For any decorated cactus  $x = (\Gamma, c_0, c_1, \dots, c_r, t)$ , let  $d(x)$  denote the number of  $v \in V_\Gamma$  which satisfies neither of the following conditions:

- $\text{val}(v) = 4$  and  $\sharp\{1 \leq i \leq r \mid v \in \lambda_\Gamma(c_i)\} = 2$ .
- $\text{val}(v) = 3$  and  $\sharp\{1 \leq i \leq r \mid v \in \lambda_\Gamma(c_i)\} = 1$ .

By induction on  $r$ , we can show that  $d(x) = 0$  if and only if  $x$  is in the image of  $\rho_B : \mathcal{P}_B \rightarrow \Lambda$ .

**Step 2.** For any  $x \in \Lambda(k : l_1, \dots, l_r)$ , there exist integers  $m_1, \dots, m_{d(x)} \geq 0$  and  $\tilde{x} \in \Lambda(k : l_1, \dots, l_r, m_1, \dots, m_{d(x)})$ , such that  $d(\tilde{x}) = 0$  and

$$x = (\cdots (\tilde{x} \circ_{r+1} \zeta_{m_1}) \circ_{r+1} \zeta_{m_2} \cdots) \circ_{r+1} \zeta_{m_{d(x)}}.$$

Since  $\tilde{x}$  is in the image of  $\rho_B$ ,  $x$  is in the image of  $\rho_{BZ}$ . Therefore,  $\rho_{BZ}$  is surjective.

The above presentation of  $x$  is unique in the following sense: for any integers  $m'_1, \dots, m'_{d(x)} \geq 0$  and  $\tilde{x}' \in \Lambda(k : l_1, \dots, l_r, m'_1, \dots, m'_{d(x)})$  such that  $d(\tilde{x}') = 0$  and

$$x = (\cdots (\tilde{x}' \circ_{r+1} \zeta_{m'_1}) \circ_{r+1} \zeta_{m'_2} \cdots) \circ_{r+1} \zeta_{m'_{d(x)}},$$

there exists  $\sigma \in \mathbb{S}_{r+d(x)}$  such that  $\sigma|_{\{1, \dots, r\}}$  is the identity,  $\tilde{x}' = \tilde{x}^\sigma$ , and  $m'_j = m_{\sigma(r+j)-r}$  for every  $j = 1, \dots, d(x)$ .

**Step 3.** For any  $X \in \mathcal{P}_B(k : l_1, \dots, l_r)$ , using relations (13) and (14), it is possible to obtain a presentation

$$X = (\beta_{m_1, i_1, n_1} \circ_1 (\beta_{m_2, i_2, n_2} \circ_1 (\cdots (\beta_{m_{r-2}, i_{r-2}, n_{r-2}} \circ_1 \beta_{m_{r-1}, i_{r-1}, n_{r-1}}) \cdots)))^\sigma,$$

where  $i_1 \geq i_2 \geq \cdots \geq i_{r-1}$ , and  $\sigma \in \mathbb{S}_r$ . This presentation is uniquely determined by  $\rho_B(X)$ . Therefore,  $\rho_B : \mathcal{P}_B \rightarrow \Lambda$  is injective. Since  $\rho_B$  factors through  $\mathcal{P}_B \rightarrow \mathcal{P}_{BZ}$ , this map is also injective. Thus, we can consider  $\mathcal{P}_B$  as a suboperad of  $\mathcal{P}_{BZ}$ .

**Step 4.** For any  $X \in \mathcal{P}_{BZ}(k : l_1, \dots, l_r)$ , there exist integers  $m_1, \dots, m_d \geq 0$  and  $\tilde{X} \in \mathcal{P}_B(k : l_1, \dots, l_r, m_1, \dots, m_d)$  such that

$$X = (\cdots ((\tilde{X} \circ_{r+1} \zeta_{m_1}) \circ_{r+1} \zeta_{m_2}) \cdots) \circ_{r+1} \zeta_{m_d}.$$

Let  $d(X)$  denote the minimal possible value of  $d$ . It is easy to see that  $d(X) \geq d(\rho_{BZ}(X))$ . We show that in fact  $d(X) = d(\rho_{BZ}(X))$ .

Suppose that  $d > d(\rho_{BZ}(X))$ . Then, either (a) or (b) holds:

- (a): There exists  $1 \leq p \leq d$  such that  $m_p = 1$ .
- (b): There exist  $1 \leq p < q \leq d$  such that  $\lambda(c_{r+p}) \cap \lambda(c_{r+q}) \neq \emptyset$ , where  $c_{r+p}, c_{r+q}$  are cycles on  $\rho_B(\tilde{X}) \in \Lambda(k : l_1, \dots, l_r, m_1, \dots, m_d)$ .



In case (a), one can use (15) to decrease  $d$  by 1. In case (b), by replacing  $\tilde{X}$  with  $\tilde{X}^\sigma$  for some  $\sigma \in \mathbb{S}_{r+d}$  if necessary, we may assume that  $p = 1$ ,  $q = 2$  and  $\rho_B(\tilde{X}) = \rho_B(Y) \circ_{r+1} \beta$ , where  $Y \in \mathcal{P}_B(k : l_1, \dots, l_r, m_1 + m_2 - 1, m_3, \dots, m_d)$  and  $\beta \in B(m_1 + m_2 - 1 : m_1, m_2)$ . Since  $\rho_B(\tilde{X}) = \rho_B(Y \circ_{r+1} \beta)$  and  $\rho_B$  is injective (Step 3), we get  $\tilde{X} = Y \circ_{r+1} \beta$ . Then,

$$(\tilde{X} \circ_{r+1} \zeta_{m_1}) \circ_{r+1} \zeta_{m_2} = Y \circ_{r+1} ((\beta \circ_1 \zeta_{m_1}) \circ_1 \zeta_{m_2}) = Y \circ_{r+1} \zeta_{m_1+m_2-1},$$

where the second equality follows from (16). Hence we can decrease  $d$  by 1. Therefore, we have shown that  $d(X) = d(\rho_{BZ}(X))$ .

**Step 5.** Suppose  $X, Y \in \mathcal{P}_{BZ}(k : l_1, \dots, l_r)$  satisfy  $\rho_{BZ}(X) = \rho_{BZ}(Y)$ . Let us set  $d := d(\rho_{BZ}(X))$ . By Step 4, there exist integers  $m_1, \dots, m_d, m'_1, \dots, m'_d \geq 0$ ,  $\tilde{X} \in \mathcal{P}_B(k : l_1, \dots, l_r, m_1, \dots, m_d)$  and  $\tilde{Y} \in \mathcal{P}_B(k : l_1, \dots, l_r, m'_1, \dots, m'_d)$ , such that

$$X = (\cdots ((\tilde{X} \circ_{r+1} \zeta_{m_1}) \circ_{r+1} \zeta_{m_2}) \cdots) \circ_{r+1} \zeta_{m_d}, \quad Y = (\cdots ((\tilde{Y} \circ_{r+1} \zeta_{m'_1}) \circ_{r+1} \zeta_{m'_2}) \cdots) \circ_{r+1} \zeta_{m'_d}.$$

By Step 2, there exists  $\sigma \in \mathbb{S}_{r+d}$ , such that  $\sigma|_{\{1, \dots, r\}}$  is the identity,  $\rho_B(\tilde{Y}) = \rho_B(\tilde{X})^\sigma$ , and  $m'_j = m_{\sigma(r+j)-r}$  for every  $j = 1, \dots, d$ . Since  $\rho_B$  is injective (Step 3),  $\tilde{Y} = \tilde{X}^\sigma$ . Hence  $X = Y$ , thus  $\rho_{BZ}$  is injective. Since we already proved that  $\rho_{BZ}$  is surjective (Step 2), we have proved that  $\rho_{BZ}$  is an isomorphism.

**Step 6.** Finally, we show that  $\rho_{BZT} : \mathcal{P}_{BZT} \rightarrow f\Lambda$  is an isomorphism. For any  $x \in f\Lambda(k : l_1, \dots, l_r)$ , there exist  $\bar{x} \in \Lambda(k : l_1, \dots, l_r)$  and integers  $i_1, \dots, i_r$ , such that  $x = (\cdots ((\bar{x} \circ_1 \tau_{l_1}^{i_1}) \circ_2 \tau_{l_2}^{i_2}) \cdots) \circ_r \tau_{l_r}^{i_r}$  and  $0 \leq i_s \leq l_s$  ( $\forall s = 1, \dots, r$ ). Moreover,  $\bar{x}, i_1, \dots, i_r$  are uniquely determined by  $x$ .

Then, surjectivity of  $\rho_{BZT}$  immediately follows from surjectivity of  $\rho_{BZ}$ . On the other hand, using relations (17), (18), (19), we can show the following: for any  $X \in \mathcal{P}_{BZT}(k : l_1, \dots, l_r)$ , there exist  $\bar{X} \in \mathcal{P}_{BZ}(k : l_1, \dots, l_r)$  and integers  $i_1, \dots, i_r$ , such that  $X = (\cdots ((\bar{X} \circ_1 \tau_{l_1}^{i_1}) \circ_2 \tau_{l_2}^{i_2}) \cdots) \circ_r \tau_{l_r}^{i_r}$ , and  $0 \leq i_s \leq l_s$  ( $\forall s = 1, \dots, r$ ). Since  $\rho_{BZ}(\bar{X})$  and  $i_1, \dots, i_r$  are uniquely determined by  $\rho_{BZT}(X)$ , injectivity of  $\rho_{BZT}$  follows from injectivity of  $\rho_{BZ}$ .  $\square$

For any set  $S$ , let  $\mathbb{R}[S]$  denote the  $\mathbb{R}$ -vector space freely generated by  $S$ , i.e.  $\mathbb{R}[S] := \bigoplus_{s \in S} \mathbb{R}s$ . For any  $K$ -colored operad of sets  $S = (S(k : l_1, \dots, l_r))_{k, l_1, \dots, l_r \in K}$ , we define a  $K$ -colored graded operad  $\mathbb{R}[S]$  by

$$\mathbb{R}[S](k : l_1, \dots, l_r)_* := \begin{cases} \mathbb{R}[S(k : l_1, \dots, l_r)] & (* = 0) \\ 0 & (* \neq 0) \end{cases}$$

with the natural composition maps and symmetric group actions. We call dg  $\mathbb{R}[S]$ -algebras simply as dg  $S$ -algebras. The next corollary is an immediate consequence of Lemma 9.3.

**Corollary 9.4.** *Let  $\mathcal{O} = (\mathcal{O}(k))_{k \geq 0}$  be a sequence of dg vector spaces.*

- (i): *The following structures on  $\mathcal{O}$  are equivalent:*
  - *A structure of a nonsymmetric dg operad with a multiplication  $\mu \in \mathcal{O}(2)_0$  and a unit  $\varepsilon \in \mathcal{O}(0)_0$ .*
  - *A structure of a dg  $\Lambda$ -algebra.*

The correspondence is given by the following formulas:

$$\begin{aligned}\beta_{k,i,l} \cdot (x \otimes y) &= x \circ_i y \quad (1 \leq i \leq k, l \geq 0, x \in \mathcal{O}(k), y \in \mathcal{O}(l)), \\ \zeta_0(1) &= \varepsilon, \quad \zeta_1(1) = 1_{\mathcal{O}}, \quad \zeta_2(1) = \mu.\end{aligned}$$

(ii): The following structures on  $\mathcal{O}$  are equivalent:

- A structure of a cyclic nonsymmetric dg operad with a multiplication  $\mu \in \mathcal{O}(2)_0$  and a unit  $\varepsilon \in \mathcal{O}(0)_0$ , such that  $\mu$  is cyclically invariant.
- A structure of a dg  $f\Lambda$ -algebra.

The correspondence is given by the formulas in (i) and  $\tau_k \cdot x = \tau_k(x)$  ( $\forall k \geq 1$ ).

**Remark 9.5.** By Corollary 9.4 (ii),  $\mathcal{CL} = (C_{*+d}^{\text{dR}}(\bar{\mathcal{L}}_{k,\text{reg}}))_{k \geq 0}$  has a dg  $f\Lambda$ -algebra structure. For any  $y \in C_*^{\text{dR}}(\mathcal{L}_{k,\text{reg}})$ , let  $|y| := \inf\{a \mid y \in C_*^{\text{dR}}(\mathcal{L}_{k,\text{reg}}^a)\}$ . Then, it is easy to see that

$$|x \cdot (y_1 \otimes \cdots \otimes y_r)| \leq |y_1| + \cdots + |y_r| \quad (\forall x \in f\Lambda(k : l_1, \dots, l_r), \forall y_i \in \mathcal{CL}(l_i) (i = 1, \dots, r)).$$

Using this fact and the definition of the dg  $f\tilde{\Lambda}$ -algebra structure on  $\tilde{\mathcal{O}}$  (see the next section), we conclude that the dg  $f\tilde{\Lambda}$ -algebra structure on  $C_*^{\mathcal{LM}}$  preserves the length filtration, as we claimed in Proposition 1.10 (ii).

## 10. DG OPERADS $f\tilde{\Lambda}$ AND $\tilde{\Lambda}$

In this section, we define dg operads  $f\tilde{\Lambda}$  and  $\tilde{\Lambda}$ , and reduce Theorem 6.7 to Theorem 10.4 and Lemma 10.3, which we prove in the next section.

For any integer  $r \geq 0$ , we define graded vector spaces  $f\tilde{\Lambda}(r)$  and  $\tilde{\Lambda}(r)$  by

$$f\tilde{\Lambda}(r)_* := \coprod_{\substack{k, l_1, \dots, l_r \in \mathbb{Z}_{\geq 0} \\ l_1 + \cdots + l_r - k = *}} \mathbb{R}[f\Lambda(k : l_1, \dots, l_r)], \quad \tilde{\Lambda}(r)_* := \coprod_{\substack{k, l_1, \dots, l_r \in \mathbb{Z}_{\geq 0} \\ l_1 + \cdots + l_r - k = *}} \mathbb{R}[\Lambda(k : l_1, \dots, l_r)].$$

Obviously,  $\tilde{\Lambda}(r)_* \subset f\tilde{\Lambda}(r)_*$ . For any  $x \in f\tilde{\Lambda}(r)_*$ , we denote its  $(k : l_1, \dots, l_r)$ -component by  $x_{k:l_1, \dots, l_r}$ .

**Remark 10.1.** As is clear from the definition, the graded vector space  $f\tilde{\Lambda}(r)_*$  is unbounded. In fact,  $f\tilde{\Lambda}(0)_N \neq 0$  for every  $N \leq 0$ . When  $r \geq 1$ ,  $f\tilde{\Lambda}(r)_N \neq 0$  for every  $N \in \mathbb{Z}$ .

We show that  $f\tilde{\Lambda} := (f\tilde{\Lambda}(r))_{r \geq 0}$  has a natural dg operad structure, and  $\tilde{\Lambda} := (\tilde{\Lambda}(r))_{r \geq 0}$  is its suboperad.

**Composition map:** For any  $1 \leq i \leq r$  and  $s \geq 0$ , we define a degree 0 linear map  $\circ_i : f\tilde{\Lambda}(r)_* \otimes f\tilde{\Lambda}(s)_* \rightarrow f\tilde{\Lambda}(r+s-1)_*$  by

$$(x \circ_i y)_{k:l_1, \dots, l_{r+s-1}} := \begin{cases} (-1)^{|y|(l_{i+s} + \cdots + l_{r+s-1})} x_{k:l_1, \dots, l_{i-1}, m, l_{i+s}, \dots, l_{r+s-1}} \circ_i y_{m:l_i, \dots, l_{i+s-1}} & (m \geq 0) \\ 0 & (m < 0), \end{cases}$$

where  $m := l_i + \cdots + l_{i+s-1} - |y|$ .

**Associativity:** For any  $x \in f\tilde{\Lambda}(r)_*$ ,  $y \in f\tilde{\Lambda}(s)_*$  and  $z \in f\tilde{\Lambda}(t)_*$ , there holds

$$\begin{aligned} (x \circ_i y) \circ_{j+s-1} z &= (-1)^{|y||z|} (x \circ_j z) \circ_i y & (1 \leq i < j \leq r), \\ x \circ_i (y \circ_j z) &= (x \circ_i y) \circ_{i+j-1} z & (1 \leq i \leq r, 1 \leq j \leq s). \end{aligned}$$

This follows from associativity of composition maps on  $f\Lambda$  and sign computations.

**Unit:** We define  $\tilde{\varepsilon} \in \tilde{\Lambda}(1)_0 = \prod_{k \geq 0} \mathbb{R}[\Lambda(k : k)]$  by  $\tilde{\varepsilon}_{k:k} := \varepsilon_k (\forall k \geq 0)$ . Then, for any  $x \in f\tilde{\Lambda}(r)_*$ , there holds  $x = \tilde{\varepsilon} \circ_1 x$  and  $x = x \circ_i \tilde{\varepsilon}$  for any  $i = 1, \dots, r$ .

**Differential:** For any integer  $k \geq 1$ , let us define  $\delta_k \in \mathbb{R}[\Lambda(k : k-1)]$  by  $\delta_k := \sum_{i=0}^k (-1)^i \delta_{k,i}$ . Then,  $\delta_{k+1} \circ_1 \delta_k = 0$  for any  $k \geq 1$ . We define  $\tilde{\delta} \in \tilde{\Lambda}(1)_{-1} = \prod_{k \geq 1} \mathbb{R}[\Lambda(k : k-1)]$  by  $\tilde{\delta}_{k:k-1} := \delta_k (\forall k \geq 1)$ . Then,  $\tilde{\delta} \circ_1 \tilde{\delta} = 0$ . We define a differential  $\partial : f\tilde{\Lambda}(r)_* \rightarrow f\tilde{\Lambda}(r)_{*-1}$  by

$$\partial x := \tilde{\delta} \circ_1 x - (-1)^{|x|} \sum_{j=1}^r x \circ_j \tilde{\delta}.$$

By associativity of composition maps, it is easy to check  $\partial^2 = 0$  and the Leibniz rule

$$\partial(x \circ_i y) = \partial x \circ_i y + (-1)^{|x|} x \circ_i \partial y \quad (x \in f\tilde{\Lambda}(r)_*, y \in f\tilde{\Lambda}(s)_*, i = 1, \dots, r).$$

**Symmetric group actions:** For any  $r \geq 0$ , we define a linear map  $f\tilde{\Lambda}(r)_* \otimes \mathbb{R}[\mathbb{S}_r] \rightarrow f\tilde{\Lambda}(r)_*$ ;  $x \otimes \sigma \mapsto x^\sigma$  by

$$(x^\sigma)_{k:l_1, \dots, l_r} := \prod_{\substack{i < j \\ \sigma(i) > \sigma(j)}} (-1)^{l_i l_j} \cdot (x_{k:l_{\sigma^{-1}(1)}, \dots, l_{\sigma^{-1}(r)}})^\sigma.$$

By direct computations on signs, we can check that this is a chain map, and compatible with composition maps.

**Suboperad  $\tilde{\Lambda}$ :** Since  $\Lambda$  is a suboperad of  $f\Lambda$ , it follows that  $\tilde{\Lambda}$  is a suboperad of  $f\tilde{\Lambda}$ .

We show that for any dg  $\Lambda$  (resp.  $f\Lambda$ ) algebra  $\mathcal{O}$ , the total complex  $\tilde{\mathcal{O}}$  has the dg  $\tilde{\Lambda}$  (resp.  $f\tilde{\Lambda}$ ) algebra structure.

**Lemma 10.2.** (i): Let  $\mathcal{O} = (\mathcal{O}(k)_*)_{k \geq 0}$  be a dg  $\Lambda$  (resp.  $f\Lambda$ ) algebra. Then,  $\tilde{\mathcal{O}}_* = \prod_{k \geq 0} \mathcal{O}(k)_{*+k}$  has the natural dg  $\tilde{\Lambda}$  (resp.  $f\tilde{\Lambda}$ ) algebra structure.

(ii): Let  $\varphi : \mathcal{O} \rightarrow \mathcal{O}'$  be a morphism of dg  $\Lambda$  (resp.  $f\Lambda$ ) algebras. Then,  $\tilde{\varphi} : \tilde{\mathcal{O}} \rightarrow \tilde{\mathcal{O}}'$  is a morphism of dg  $\tilde{\Lambda}$  (resp.  $f\tilde{\Lambda}$ )-algebras.

**Proof.** Let  $\mathcal{O} = (\mathcal{O}(k))_{k \geq 0}$  be a dg  $\Lambda$ -algebra. Then, for any  $k, l_1, \dots, l_r \geq 0$ , we have a chain map

$$\mathbb{R}[\Lambda](k : l_1, \dots, l_r) \otimes \mathcal{O}(l_1) \otimes \dots \otimes \mathcal{O}(l_r) \rightarrow \mathcal{O}(k); x \otimes y_1 \otimes \dots \otimes y_r \mapsto x \cdot (y_1 \otimes \dots \otimes y_r).$$

For every  $r \geq 0$ , we define  $\tilde{\Lambda}(r) \otimes \tilde{\mathcal{O}}^{\otimes r} \rightarrow \tilde{\mathcal{O}} : x \otimes y^1 \otimes \cdots \otimes y^r \mapsto x \cdot (y^1 \otimes \cdots \otimes y^r)$  by

$$(x \cdot (y^1 \otimes \cdots \otimes y^r))_k := \sum_{l_1 + \cdots + l_r = k + |x|} (-1)^{\dagger} x_{k:l_1, \dots, l_r} \cdot (y_{l_1}^1 \otimes \cdots \otimes y_{l_r}^r),$$

$$\dagger := (k+1)|x| + \sum_{i=1}^r (k + l_i + \cdots + l_r) |y^i|.$$

Notice that the RHS of the first formula is a finite sum. Direct computations show that this is a dg  $\tilde{\Lambda}$ -algebra structure on  $\tilde{\mathcal{O}}$ . Thus we have verified Lemma 10.2 (i) for  $\Lambda$ . (ii) is straightforward from the construction. The case for  $f\Lambda$  is completely parallel, and omitted.  $\square$

Let us define  $\alpha \in \tilde{\Lambda}(2)_0$ ,  $\beta \in \tilde{\Lambda}(2)_1$ ,  $\rho \in f\tilde{\Lambda}(1)_1$  by the following formulas:

$$\alpha_k := \sum_{l+m=k} (-1)^{lm} \alpha_{l,m}$$

$$\beta_k := \sum_{\substack{l+m=k+1 \\ 1 \leq i \leq l}} (-1)^{im} \beta_{l,i,m} + (-1)^{(i+l)m} (\beta_{l,i,m})^{(12)},$$

$$\rho_k := \sum_{0 \leq i \leq k} (-1)^{(i+1)k} \sigma_{k,i} \circ_1 \tau_{k+1}^{k+1-i}.$$

Let us define operators  $\bullet$ ,  $\{, \}$  and  $\Delta$  on  $\tilde{\mathcal{O}}$  by

$$v \bullet w := \alpha \cdot (v \otimes w), \quad \{v, w\} := (-1)^{|v|} \beta \cdot (v \otimes w), \quad \Delta v := \rho \cdot v.$$

Now, Theorem 6.7 follows from the next two results, which are proved in the next section.

**Lemma 10.3.** (i):  $\alpha$  satisfies  $\partial\alpha = 0$  and  $\alpha \circ_1 \alpha = \alpha \circ_2 \alpha$ . In particular,  $\bullet$  defines a dga algebra structure on  $\tilde{\mathcal{O}}$ .

(ii):  $\beta$  satisfies  $\partial\beta = 0$  and  $\beta^{(12)} = \beta$ ,  $\beta \circ_1 \beta + (\beta \circ_1 \beta)^{(123)} + (\beta \circ_1 \beta)^{(321)} = 0$ . In particular,  $\{, \}$  defines a dg Lie algebra structure on  $\tilde{\mathcal{O}}$ .

(iii):  $\rho$  satisfies  $\partial\rho = 0$ . In particular,  $\Delta$  is an anti-chain map on  $\tilde{\mathcal{O}}$ .

**Theorem 10.4.** There exist isomorphisms  $H_*(\tilde{\Lambda}) \cong \mathcal{G}$  and  $H_*(f\tilde{\Lambda}) \cong \mathcal{BV}$ , compatible with the inclusion maps. Moreover,  $[\alpha] \in H_0(\tilde{\Lambda}(2))$ ,  $[\beta] \in H_1(\tilde{\Lambda}(2))$ ,  $[\rho] \in H_1(f\tilde{\Lambda}(1))$  correspond to  $a \in \mathcal{G}(2)_0$ ,  $b \in \mathcal{G}(2)_1$ ,  $\Delta \in \mathcal{BV}(1)_1$  via these isomorphisms.

## 11. HOMOLOGY OF $f\tilde{\Lambda}$ AND $\tilde{\Lambda}$

The goal of this section is to prove Theorem 10.4. In the course of the proof, we also verify Lemma 10.3 (see Remark 11.6). Here we explain an outline of our proof assuming Lemmas 11.1, 11.2, 11.3. For any  $r \geq 0$  and  $k \geq 0$ , we define

$$f\tilde{\Lambda}_{\geq k}(r)_* := \prod_{\substack{k' \in \mathbb{Z}_{\geq k}, l_1, \dots, l_r \in \mathbb{Z}_{\geq 0} \\ l_1 + \cdots + l_r - k' = *}} \mathbb{R}[f\Lambda(k' : l_1, \dots, l_r)].$$

Obviously  $f\tilde{\Lambda}_{\geq 0}(r) = f\tilde{\Lambda}(r)$ , and  $f\tilde{\Lambda}_{\geq k}(r)$  is a subcomplex of  $f\tilde{\Lambda}(r)$  for any  $k \geq 0$ . We denote the quotient  $f\tilde{\Lambda}_{\geq k}(r)/f\tilde{\Lambda}_{\geq k+1}(r)$  by  $f\tilde{\Lambda}_k(r)$ .  $\tilde{\Lambda}_{\geq k}(r)$  and  $\tilde{\Lambda}_k(r)$  are defined in the same way. Let us consider quotient maps

$$Q : f\tilde{\Lambda}(r)_* \rightarrow f\tilde{\Lambda}_0(r)_*, \quad Q|_{\tilde{\Lambda}} : \tilde{\Lambda}(r)_* \rightarrow \tilde{\Lambda}_0(r)_*.$$

Our first step is to prove the following lemma in Section 11.1.

**Lemma 11.1.** *The quotient maps  $Q$  and  $Q|_{\tilde{\Lambda}}$  are quasi-isomorphisms.*

Our next step is to define operad structures on  $f\tilde{\Lambda}_0 := (f\tilde{\Lambda}_0(r))_{r \geq 0}$  and  $\tilde{\Lambda}_0 := (\tilde{\Lambda}_0(r))_{r \geq 0}$ , and compare them with  $f\tilde{\Lambda}$  and  $\tilde{\Lambda}$ . The next lemma is proved in Section 11.2.

**Lemma 11.2.**  *$f\tilde{\Lambda}_0$  has a dg operad structure, such that  $\tilde{\Lambda}_0$  is its suboperad. There exists a morphism of dg operads  $P : f\tilde{\Lambda}_0 \rightarrow f\tilde{\Lambda}$ , such that  $P(\tilde{\Lambda}_0) \subset \tilde{\Lambda}$  and  $Q \circ P = \text{id}_{f\tilde{\Lambda}_0}$ . In particular,  $P$  and  $P|_{\tilde{\Lambda}_0} : \tilde{\Lambda}_0 \rightarrow \tilde{\Lambda}$  are quasi-isomorphisms of dg operads.*

Now, the proof of  $H_*(f\tilde{\Lambda}) \cong \mathcal{BV}$  and  $H_*(\tilde{\Lambda}) \cong \mathcal{G}$  is reduced to the following lemma.

**Lemma 11.3.** *There exist isomorphisms  $H_*(f\tilde{\Lambda}_0) \cong \mathcal{BV}$  and  $H_*(\tilde{\Lambda}_0) \cong \mathcal{G}$ , compatible with the inclusion maps.*

Lemma 11.3 is proved in Sections 11.3 and 11.4.

**11.1. Proof of Lemma 11.1.** We only prove that  $Q|_{\tilde{\Lambda}} : \tilde{\Lambda}(r)_* \rightarrow \tilde{\Lambda}_0(r)_*$  is a quasi-isomorphism. The proof for  $Q : f\tilde{\Lambda}(r)_* \rightarrow f\tilde{\Lambda}_0(r)_*$  is completely parallel.

Since  $Q|_{\tilde{\Lambda}}$  is surjective, it is enough to show that  $\ker Q|_{\tilde{\Lambda}} = \tilde{\Lambda}_{\geq 1}(r)_*$  is acyclic. For each integer  $k \geq 0$ , we denote the differential on  $\tilde{\Lambda}_k(r)$  by  $\partial_k$ . In other words,  $\partial_k x := (-1)^{|x|+1} \sum_{j=1}^r x \circ_j \tilde{\delta}$ . Let us define an anti-chain map  $D_k : \tilde{\Lambda}_k(r)_* \rightarrow \tilde{\Lambda}_{k+1}(r)_{*-1}$  by  $D_k(x) := \delta_{k+1} \circ_1 x$ . Then, the chain complex  $\tilde{\Lambda}_{\geq 1}(r)_*$  is identified with  $\prod_{k \geq 1} \tilde{\Lambda}_k(r)_*$ , where the differential is  $\partial + D : (x_k)_k \mapsto (\partial_k x_k)_k + (D_k(x_k))_{k+1}$ . Now, let us assume the following lemma.

**Lemma 11.4.** *For every integer  $k \geq 1$ , there exists a linear map  $J_k : \tilde{\Lambda}_k(r)_* \rightarrow \tilde{\Lambda}_0(r)_{*+k}$  which satisfies the following properties:*

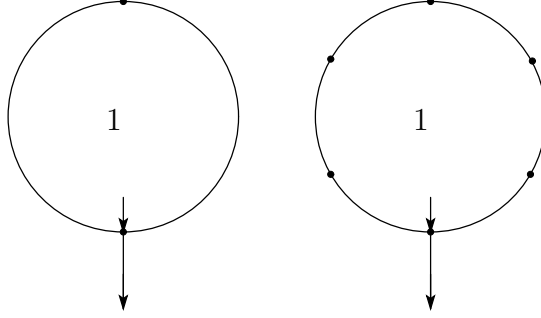
- $J_k$  is a chain (resp. anti-chain) map, if  $k$  is even (resp. odd).
- For every  $k$ ,  $J_k$  induces an isomorphism on homology.
- For every  $k$ , there holds  $J_{k+1} \circ D_k = \alpha_k \cdot J_k$ , where  $\alpha_k := \sum_{i=0}^{k+1} (-1)^i$ .

By Lemma 11.4, the sequence

$$0 \longrightarrow H_*(\tilde{\Lambda}_1(r)) \xrightarrow{H_*(D_1)} H_{*-1}(\tilde{\Lambda}_2(r)) \xrightarrow{H_*(D_2)} H_{*-2}(\tilde{\Lambda}_3(r)) \xrightarrow{H_*(D_3)} \cdots$$

is exact. Hence, Lemma 6.1 shows that  $\text{Ker}Q|_{\tilde{\Lambda}} = \tilde{\Lambda}_{\geq 1}(r)_*$  is acyclic. Therefore, it is enough to prove Lemma 11.4. The proof consists of four steps.

**Step 1.** Let us define  $j_k \in \Lambda(0 : k)$  by  $\Gamma_{j_k} := \Gamma_{\varepsilon_k} \setminus \{t_1, \dots, t_k\}$ ,  $c_0^{j_k} := c_0^{\varepsilon_k} \setminus \{t_1, \dots, t_k\}$ , and  $c_1^{j_k} := c_1^{\varepsilon_k}$ ,  $t_{j_k} := t_{\varepsilon_k}$ . (Recall the definition of  $\varepsilon_k$  in Section 8.4.)



$j_1$  (left) and  $j_5$  (right)

We define  $J_k : \tilde{\Lambda}_k(r)_* \rightarrow \tilde{\Lambda}_0(r)_{*+k}$  by  $J_k(x) := j_k \circ_1 x$ . (Briefly speaking,  $J_k$  removes all tails other than  $t$ .) Then, there holds

$$J_k(\partial_k x) = (-1)^{|x|+1} \sum_{j=1}^r j_k \circ_1 x \circ_j \tilde{\delta}, \quad \partial_0(J_k x) = (-1)^{|x|+k+1} \sum_{j=1}^r j_k \circ_1 x \circ_j \tilde{\delta}.$$

Hence,  $J_k$  is a chain (resp. anti-chain) map if  $k$  is even (resp. odd). On the other hand, since  $j_{k+1} \circ_1 \delta_{k+1} = \alpha_k \cdot j_k$ , there holds  $J_{k+1} \circ D_k = \alpha_k \cdot J_k$ .

**Step 2.** To show that  $J_k$  induces an isomorphism on homology, we define a filtration on the chain complex  $\tilde{\Lambda}_k(r)_*$ . For any  $x = (\Gamma, c_0, \dots, c_r, t) \in \Lambda(k : l_1, \dots, l_r)$ , we define positive integers  $w_1(x) < \dots < w_k(x)$  so that  $(N_\Gamma \circ \iota_\Gamma)^{w_j(x)}(t) \in T_\Gamma$ . For every  $m = 0, \dots, k$ , we define

$$F_m \Lambda(k : l_1, \dots, l_r) := \{x \in \Lambda(k : l_1, \dots, l_r) \mid w_j(x) = j \ (\forall j = 1, \dots, m)\},$$

$$F_m \tilde{\Lambda}_k(r)_* := \prod_{l_1 + \dots + l_r - k = *} \mathbb{R}[F_m \Lambda(k : l_1, \dots, l_r)].$$

$F_m \tilde{\Lambda}_k(r)$  is a subcomplex of  $\tilde{\Lambda}_k(r)$  for every  $m \geq 0$ , and we obtain a filtration  $\tilde{\Lambda}_k(r) = F_0 \tilde{\Lambda}_k(r) \supset F_1 \tilde{\Lambda}_k(r) \supset \dots \supset F_k \tilde{\Lambda}_k(r)$ . Since  $F_k \Lambda(k : l_1, \dots, l_r) \rightarrow \Lambda(0 : l_1, \dots, l_r); x \mapsto j_k \circ_1 x$  is a bijection,  $J_k|_{F_k \tilde{\Lambda}_k} : F_k \tilde{\Lambda}_k(r)_* \rightarrow \tilde{\Lambda}_0(r)_{*+k}$  is an isomorphism. Therefore, to show that  $J_k$  is a quasi-isomorphism, it is enough to show that  $F_m \tilde{\Lambda}_k(r)/F_{m+1} \tilde{\Lambda}_k(r)$  is acyclic for every  $m = 0, \dots, k-1$ .

**Step 3.** For every  $n \geq 1$ , we define

$$F_{m,n} \Lambda(k : l_1, \dots, l_r) := \{x \in F_m \Lambda(k : l_1, \dots, l_r) \mid w_{m+1}(x) \leq m+n\},$$

$$F_{m,n} \tilde{\Lambda}_k(r)_* := \prod_{l_1 + \dots + l_r - k = *} \mathbb{R}[F_{m,n} \Lambda(k : l_1, \dots, l_r)].$$

Notice that

$$F_{m,1}\Lambda(k : l_1, \dots, l_r) = F_{m+1}(k : l_1, \dots, l_r),$$

$$n > r + \sum_{j=1}^r l_j \implies F_{m,n}\Lambda(k : l_1, \dots, l_r) = F_m(k : l_1, \dots, l_r).$$

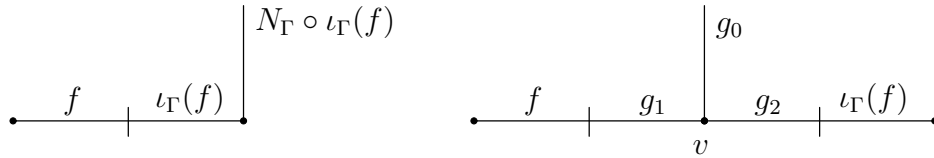
Hence,  $F_m\tilde{\Lambda}_k(r)/F_{m+1}\tilde{\Lambda}_k(r) = \lim_{n \rightarrow \infty} F_{m,n}\tilde{\Lambda}_k(r)/F_{m,1}\tilde{\Lambda}_k(r)$ . Therefore, it is enough to show that  $F_{m,n+1}\tilde{\Lambda}_k(r)/F_{m,n}\tilde{\Lambda}_k(r)$  is acyclic for every  $n \geq 1$ .

**Step 4.** We define a degree 1 linear map  $K$  on  $F_{m,n+1}\tilde{\Lambda}_k(r)/F_{m,n}\tilde{\Lambda}_k(r)$ . Let  $x = (\Gamma, c_0, \dots, c_r, t) \in F_{m,n+1}\Lambda(k : l_1, \dots, l_r) \setminus F_{m,n}\Lambda(k : l_1, \dots, l_r)$ . Then  $w_j(x) = j$  for  $j = 1, \dots, m$ , and  $w_{m+1}(x) = m + n + 1$ . Let  $f := (N_\Gamma \circ \iota_\Gamma)^{m+n}(t) \in c_0$ . Then,  $f \notin T_\Gamma$  since  $w_m(x) = m < m + n < w_{m+1}(x)$ . Let us take  $i \in \{1, \dots, r\}$  so that  $\iota_\Gamma(f) \in c_i$ , and define  $x' = (\Gamma', c'_0, \dots, c'_r, t)$  as follows:

$$V_{\Gamma'} := V_\Gamma \cup \{v\}, \quad F_{\Gamma'} := F_\Gamma \setminus \{N_\Gamma \circ \iota_\Gamma(f)\} \cup \{g_0, g_1, g_2\},$$

$$\iota_{\Gamma'}(h) := \begin{cases} g_0 & (h = g_0) \\ g_1 & (h = f) \\ g_2 & (h = \iota_\Gamma(f)) \\ f & (h = g_1) \\ \iota_\Gamma(f) & (h = g_2) \\ \iota_\Gamma(h) & (\text{otherwise}), \end{cases} \quad N_{\Gamma'}(h) := \begin{cases} g_0 & (h = g_1) \\ g_1 & (h = g_2) \\ g_2 & (h = g_0) \\ N_\Gamma^2 \circ \iota_\Gamma(f) & (h = \iota_\Gamma(f)) \\ N_\Gamma(h) & (\text{otherwise}), \end{cases}$$

$$\lambda_{\Gamma'}(h) := \begin{cases} v & (h = g_0, g_1, g_2) \\ \lambda_\Gamma(h) & (\text{otherwise}), \end{cases} \quad c'_j := \begin{cases} c_0 \setminus \{N_\Gamma \circ \iota_\Gamma(f)\} \cup \{g_0, g_2\} & (j = 0) \\ c_i \cup \{g_1\} & (j = i) \\ c_j & (\text{otherwise}). \end{cases}$$



$x$  (left) and  $x'$  (right)

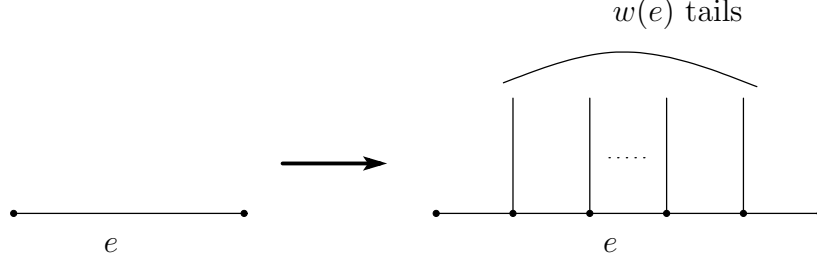
Then,  $x' \in F_{m,n+1}\Lambda(k : l_1, \dots, l_i + 1, \dots, l_r) \setminus F_{m,n}\Lambda(k : l_1, \dots, l_i + 1, \dots, l_r)$ .

We define  $K$  by  $K(x) := (-1)^{k+l_1+\dots+l_{i-1}+j+1} \cdot x'$ , where  $j \in \{0, \dots, l_i\}$  is defined by  $\iota_\Gamma(f) = (N_\Gamma \circ \iota_\Gamma)^j(\text{fr}(c_i))$ . Simple computations show  $\partial K + K\partial = \text{id}$ , therefore  $F_{m,n+1}\tilde{\Lambda}_k(r)/F_{m,n}\tilde{\Lambda}_k(r)$  is acyclic. This completes the proof of Lemma 11.4.

**11.2. Proof of Lemma 11.2.** Let us define a linear map  $P : f\tilde{\Lambda}_0(r)_* \rightarrow f\tilde{\Lambda}(r)_*$  for every integer  $r \geq 0$ . When  $r = 0$ , notice that  $f\tilde{\Lambda}_0(0)_* = \begin{cases} \mathbb{R}[\zeta_0] & (* = 0) \\ 0 & (* \neq 0). \end{cases}$  Then, we

define  $P(\zeta_0)_k := \begin{cases} \zeta_0 & (k = 0) \\ 0 & (k \geq 1). \end{cases}$

Let us consider the case  $r \geq 1$ . For any  $x = (\Gamma, c_0, \dots, c_r, t, \text{fr}) \in f\Lambda(0 : l_1, \dots, l_r)$  and  $w : E_\Gamma \rightarrow \mathbb{Z}_{\geq 0}$ , let  $P_w(x)$  denote the framed decorated cactus which is obtained by putting  $w(e)$  tails on each edge  $e \in E_\Gamma$ .



Then,  $P(x) \in f\tilde{\Lambda}(r)_*$  is defined by

$$P(x)_k := \sum_{\substack{w: E_\Gamma \rightarrow \mathbb{Z}_{\geq 0} \\ \sum_{e \in E_\Gamma} w(e) = k}} (-1)^{\sigma(x, w)} P_w(x)$$

for any  $k \geq 0$ , where  $\sigma(x, w) \in \mathbb{Z}/2\mathbb{Z}$  is defined below.

Let us abbreviate  $E_\Gamma$  as  $E$ . We define two total orders  $<_{\text{in}}$  and  $<_{\text{out}}$  on  $E$ . Notice that  $E$  is naturally identified with  $c_0 \setminus \{t\}$  and  $c_1 \cup \dots \cup c_r$ .

- Let us define an order on  $c_0 \setminus \{t\}$  by  $(N \circ \iota)(t) < \dots < (N \circ \iota)^{\#E}(t)$ , and denote the corresponding order on  $E$  as  $<_{\text{out}}$ .
- Let us define an order on  $c_1 \cup \dots \cup c_r$  by
  - When  $f \in c_i$ ,  $f' \in c_j$  and  $i < j$ , then  $f < f'$ .
  - On each  $c_i$ ,  $\text{fr}(c_i) < (N \circ \iota)(\text{fr}(c_i)) < \dots < (N \circ \iota)^{l_i}(\text{fr}(c_i))$ .

Let us denote the corresponding order on  $E$  as  $<_{\text{in}}$ .

Let  $E'$  denote the subset of  $E$  which corresponds to  $c_1 \cup \dots \cup c_r \setminus \{\text{fr}(c_1), \dots, \text{fr}(c_r)\}$  via the bijection  $E \rightarrow c_1 \cup \dots \cup c_r$ . We also define  $E'' := \{(e, e') \in E \times E \mid e <_{\text{in}} e', \quad e <_{\text{out}} e'\}$ . Then,  $\sigma(x, w) \in \mathbb{Z}/2\mathbb{Z}$  is defined as

$$\sigma(x, w) := \sum_{(e, e') \in E''} w(e)w(e') + \sum_{e \in E} w(e) \cdot \#\{e' \in E' \mid e' \leq_{\text{in}} e\}.$$

**Example 11.5.** For every  $k \geq 0$ , there holds

$$\begin{aligned} P(\varepsilon_0)_k &= \varepsilon_k, \\ P(\alpha_{0,0})_k &= \sum_{l+m=k} (-1)^{lm} \alpha_{l,m}, \\ P(\beta_{1,1,0})_k &= \sum_{\substack{l+m=k+1 \\ 1 \leq i \leq l}} (-1)^{im+i-1} \beta_{l,i,m}, \\ P(\sigma_{0,0} \circ_1 \tau_1)_k &= \sum_{0 \leq i \leq k} (-1)^{(i+1)k} \sigma_{k,i} \circ_1 \tau_{k+1}^{k+1-i}. \end{aligned}$$

**Remark 11.6.** The above formulas imply

$$P(\alpha_{0,0}) = \alpha, \quad P(\beta_{1,1,0} + \beta_{1,1,0}^{(12)}) = -\beta, \quad P(\sigma_{0,0} \circ_1 \tau_1) = \rho.$$

Then, little computations verify Lemma 10.3.



The following assertions can be checked by direct computations:

- $P$  is a chain map. Namely, for any  $x \in f\tilde{\Lambda}_0(r)_*$ , there holds

$$P\left((-1)^{|x|+1} \sum_{j=1}^r x \circ_j \tilde{\delta}\right) = \tilde{\delta} \circ_1 P(x) - (-1)^{|x|} \sum_{j=1}^r P(x) \circ_j \tilde{\delta}.$$

- For any  $x \in f\tilde{\Lambda}_0(r)_*$ ,  $1 \leq i \leq r$  and  $y \in f\tilde{\Lambda}_0(s)_*$ , there holds  $P(x \circ_i Py) = Px \circ_i Py$ .
- For any  $x \in f\tilde{\Lambda}_0(r)_*$  and  $\sigma \in \mathbb{S}_r$ , there holds  $P(x^\sigma) = P(x)^\sigma$ .

For any  $1 \leq i \leq r$  and  $s \geq 0$ , let us define a composition map on  $f\tilde{\Lambda}_0$  by

$$\circ_i : f\tilde{\Lambda}_0(r)_* \otimes f\tilde{\Lambda}_0(s)_* \rightarrow f\tilde{\Lambda}_0(r+s-1)_*; \quad x \otimes y \mapsto x \circ_i Py.$$

By the above assertions,  $f\tilde{\Lambda}_0$  is a dg operad with these composition maps (unit is  $\varepsilon_0$ ), and  $P : f\tilde{\Lambda}_0 \rightarrow f\tilde{\Lambda}$  is a morphism of dg operads. As is obvious from definitions,  $Q \circ P$  is the identity on  $f\tilde{\Lambda}_0$ , and  $P(\tilde{\Lambda}_0) \subset \tilde{\Lambda}$ . This completes the proof of Lemma 11.2.

**11.3. Proof of Lemma 11.3 modulo Lemma 11.8.** In this subsection, we reduce Lemma 11.3 to Lemma 11.8.

Let  $x = (\Gamma, c_0, \dots, c_r, t, \text{fr}) \in f\Lambda(0 : l_1, \dots, l_r)$ .  $v \in V_\Gamma$  is called a *free vertex*, if  $\text{val}(v) = 2$ , and  $v \neq \lambda_\Gamma(\text{fr}(c_i))$  for any  $i = 1, \dots, r$ . We call  $x$  *degenerate*, if there exists a free vertex in  $V_\Gamma$ . Let  $f\Lambda^{\text{deg}}(0 : l_1, \dots, l_r)$  denote the set of degenerate elements in  $f\Lambda(0 : l_1, \dots, l_r)$ . We also set (nd stands for nondegenerate):

$$\begin{aligned} f\Lambda^{\text{nd}}(0 : l_1, \dots, l_r) &:= f\Lambda(0 : l_1, \dots, l_r) \setminus f\Lambda^{\text{deg}}(0 : l_1, \dots, l_r), \\ \Lambda^{\text{deg}}(0 : l_1, \dots, l_r) &:= f\Lambda^{\text{deg}}(0 : l_1, \dots, l_r) \cap \Lambda(0 : l_1, \dots, l_r), \\ \Lambda^{\text{nd}}(0 : l_1, \dots, l_r) &:= f\Lambda^{\text{nd}}(0 : l_1, \dots, l_r) \cap \Lambda(0 : l_1, \dots, l_r). \end{aligned}$$

Since  $\zeta_0 \in \Lambda(0 : )$  is nondegenerate, there holds

$$f\Lambda^{\text{nd}}(0 : ) = \Lambda^{\text{nd}}(0 : ) = \{\zeta_0\}, \quad f\Lambda^{\text{deg}}(0 : ) = \Lambda^{\text{deg}}(0 : ) = \emptyset.$$

For every  $r \geq 0$ , let us define

$$f\tilde{\Lambda}_0^{\text{deg}}(r)_* := \prod_{l_1 + \dots + l_r = *} \mathbb{R}[f\Lambda^{\text{deg}}(0 : l_1, \dots, l_r)], \quad \tilde{\Lambda}_0^{\text{deg}}(r)_* := \prod_{l_1 + \dots + l_r = *} \mathbb{R}[\Lambda^{\text{deg}}(0 : l_1, \dots, l_r)].$$

Notice that  $f\tilde{\Lambda}_0^{\text{deg}}(0) = \tilde{\Lambda}_0^{\text{deg}}(0) = 0$ . It is easy to see that,  $f\tilde{\Lambda}_0^{\text{deg}} := (f\tilde{\Lambda}_0^{\text{deg}}(r))_{r \geq 0}$  and  $\tilde{\Lambda}_0^{\text{deg}} := (\tilde{\Lambda}_0^{\text{deg}}(r))_{r \geq 0}$  are dg ideals of  $f\tilde{\Lambda}_0$  and  $\tilde{\Lambda}_0$ , respectively.

For any  $r \geq 1$  and  $x \in f\Lambda(0 : l_1, \dots, l_r)$ , let  $\underline{x} \in f\Lambda^{\text{nd}}(0 : l'_1, \dots, l'_r)$  denote the framed decorated cactus obtained by removing all free vertices of  $x$ . Let  $w(x) := l'_1 + \dots + l'_r + r$  denote the number of edges of  $\underline{x}$ . It is easy to see that  $w(x) \leq 3r - 1$  for any  $x \in f\Lambda(0 : l_1, \dots, l_r)$ , and  $w(x) \leq 2r - 1$  for any  $x \in \Lambda(0 : l_1, \dots, l_r)$ .

**Lemma 11.7.** *For every  $r \geq 0$ , chain complexes  $f\tilde{\Lambda}_0^{\text{deg}}(r)_*$ ,  $\tilde{\Lambda}_0^{\text{deg}}(r)_*$  are acyclic.*

**Proof.** The lemma is obvious for  $r = 0$ , thus we may assume  $r \geq 1$ . We prove the lemma only for  $\tilde{\Lambda}_0^{\text{deg}}(r)_*$ , since the proof for  $f\tilde{\Lambda}_0^{\text{deg}}(r)_*$  is similar. For  $m = 0, \dots, r - 1$ , let

$F_m \tilde{\Lambda}_0^{\deg}(r)_*$  be the subspace of  $\tilde{\Lambda}_0^{\deg}(r)_*$ , which is generated by  $x \in \Lambda^{\deg}(0 : l_1, \dots, l_r)$  such that  $w(x) \leq r + m$ . It is easy to see that this is a subcomplex of  $\tilde{\Lambda}_0^{\deg}(r)_*$ , and we obtain

$$0 = F_{-1} \tilde{\Lambda}_0^{\deg}(r)_* \subset F_0 \tilde{\Lambda}_0^{\deg}(r)_* \subset F_1 \tilde{\Lambda}_0^{\deg}(r)_* \subset \dots \subset F_{r-1} \tilde{\Lambda}_0^{\deg}(r)_* = \tilde{\Lambda}_0^{\deg}(r)_*.$$

To prove that  $\tilde{\Lambda}_0^{\deg}(r)_*$  is acyclic, it is sufficient to show that  $F_m \tilde{\Lambda}_0^{\deg}(r)_* / F_{m-1} \tilde{\Lambda}_0^{\deg}(r)_*$  is acyclic for every  $m = 0, \dots, r-1$ .

For any  $y \in \Lambda^{\text{nd}}(0 : l_1, \dots, l_r)$  such that  $l_1 + \dots + l_r = m$ , let  $V_*^y$  denote the subcomplex of  $F_m \tilde{\Lambda}_0^{\deg}(r)_* / F_{m-1} \tilde{\Lambda}_0^{\deg}(r)_*$ , which is generated by  $\{x \in \Lambda^{\deg}(0 : l'_1, \dots, l'_r) \mid \underline{x} = y\}$ . It is easy to see that  $V_*^y$  is acyclic for every  $y$ . Therefore,  $F_m \tilde{\Lambda}_0^{\deg}(r)_* / F_{m-1} \tilde{\Lambda}_0^{\deg}(r)_* =$

$$\bigoplus_{\substack{y \in \Lambda^{\text{nd}}(0 : l_1, \dots, l_r) \\ l_1 + \dots + l_r = m}} V_*^y \text{ is also acyclic.} \quad \square$$

Since  $f \tilde{\Lambda}_0^{\deg}$  and  $\tilde{\Lambda}_0^{\deg}$  are dg ideals of  $f \tilde{\Lambda}_0$  and  $\tilde{\Lambda}_0$ , the quotients  $f \tilde{\Lambda}_0^{\text{nd}} := f \tilde{\Lambda}_0 / f \tilde{\Lambda}_0^{\deg}$  and  $\tilde{\Lambda}_0^{\text{nd}} := \tilde{\Lambda}_0 / \tilde{\Lambda}_0^{\deg}$  are dg operads. As graded vector spaces,

$$f \tilde{\Lambda}_0^{\text{nd}}(r)_* = \prod_{l_1 + \dots + l_r = *} \mathbb{R}[f \Lambda^{\text{nd}}(0 : l_1, \dots, l_r)], \quad \tilde{\Lambda}_0^{\text{nd}}(r)_* = \prod_{l_1 + \dots + l_r = *} \mathbb{R}[\Lambda^{\text{nd}}(0 : l_1, \dots, l_r)].$$

By Lemma 11.7, the morphisms of dg operads  $f \tilde{\Lambda}_0 \rightarrow f \tilde{\Lambda}_0^{\text{nd}}$  and  $\tilde{\Lambda}_0 \rightarrow \tilde{\Lambda}_0^{\text{nd}}$  are quasi-isomorphisms. On the other hand, one can define morphisms of graded operads  $\Phi^{\mathcal{G}} : \mathcal{G} \rightarrow H_*(\tilde{\Lambda}_0^{\text{nd}})$  and  $\Phi^{\mathcal{BV}} : \mathcal{BV} \rightarrow H_*(f \tilde{\Lambda}_0^{\text{nd}})$  by

$$u \mapsto [\zeta_0], \quad a \mapsto [\alpha_{0,0}], \quad b \mapsto -[\beta_{1,1,0} + \beta_{1,1,0}^{(12)}], \quad \Delta \mapsto [\sigma_{0,0} \circ_1 \tau_1].$$

The relevant relations are checked by direct computations. Now, Lemma 11.3 is reduced to the following lemma.

**Lemma 11.8.**  $\Phi^{\mathcal{G}}$  and  $\Phi^{\mathcal{BV}}$  are isomorphisms.

**Remark 11.9.** It seems that the dg operad  $\tilde{\Lambda}_0^{\text{nd}}$  (resp.  $f \tilde{\Lambda}_0^{\text{nd}}$ ) is isomorphic (up to sign) to the dg operad  $CC_*(\text{Cact}^1)$  (resp.  $CC_*(\text{Cacti}^1)$ ), which is introduced in [21] (resp. [22]). These dg operads are defined by CW-decompositions of the cacti and spineless cacti operads. In papers [20], [21], [22], it is proved that these operads are chain models for the (framed) little disks operad, therefore their homology are isomorphic to the Gerstenhaber and BV operads. Lemma 11.8 may follow from these results, however we give a direct proof in the next subsection.

**11.4. Proof of Lemma 11.8.** In this subsection, we abbreviate  $\tilde{\Lambda}_0^{\text{nd}}(r)$  and  $f \tilde{\Lambda}_0^{\text{nd}}(r)$  by  $\Lambda(r)$  and  $f \Lambda(r)$  for simplicity. For every integer  $r \geq 1$ , we define a chain complex  $\Gamma_*^r$  by

$$\Gamma_0^r := \mathbb{R}\gamma^0, \quad \Gamma_1^r := \bigoplus_{i=1}^r \mathbb{R}\gamma_i^1, \quad \Gamma_*^r = 0 \ (* \neq 0, 1),$$

and differential is zero. We define a chain map  $\Psi^r : \Lambda(r)_* \otimes \Gamma_*^r \rightarrow \Lambda(r+1)_*$  by

$$\Psi^r(x \otimes \gamma^0) := x \circ_r \alpha_{0,0}, \quad \Psi^r(x \otimes \gamma_i^1) := (x \circ_i (\beta_{1,1,0} + \beta_{1,1,0}^{(12)}))^{\sigma_i},$$

where  $\sigma_i \in \mathbb{S}_{r+1}$  is defined as  $\sigma_i(j) := \begin{cases} j & (j \leq i) \\ j+1 & (j = i+1, \dots, r) \\ i+1 & (j = r+1). \end{cases}$

**Lemma 11.10.** *For every  $r \geq 1$ ,  $\Psi^r$  is a quasi-isomorphism.*

**Proof.** For every  $m \geq 0$ , let  $F_m\Lambda(r)$  denote the subspace of  $\Lambda(r)$ , which is generated by  $x \in \Lambda^{\text{nd}}(0 : l_1, \dots, l_r)$ , such that  $l_1 + \dots + l_r \leq m$ . This is a subcomplex of  $\Lambda(r)$ . We set  $F_{-1}\Lambda(r) = 0$ .

On the other hand, let  $F_m\Lambda(r+1)$  denote the subspace of  $\Lambda(r+1)$ , which is generated by  $x \in \Lambda^{\text{nd}}(0 : l_1, \dots, l_{r+1})$  such that  $w(x \circ_{r+1} \zeta_{l_{r+1}}) \leq r+m$ . For every  $m \geq 0$ , this is a subcomplex of  $\Lambda(r+1)$ . We set  $F_{-1}\Lambda(r+1) = 0$ .

It is easy to see that  $\Psi^r(F_m\Lambda(r) \otimes \Gamma^r) \subset F_m\Lambda(r+1)$  for every  $m \geq 0$ . Thus,  $\Psi^r$  induces a map on quotients

$$(20) \quad (F_m\Lambda(r)/F_{m-1}\Lambda(r)) \otimes \Gamma^r \rightarrow F_m\Lambda(r+1)/F_{m-1}\Lambda(r+1).$$

To show that  $\Psi^r$  is a quasi-isomorphism, it is enough to show that (20) is a quasi-isomorphism for every  $m \geq 0$ .

$F_m\Lambda(r)/F_{m-1}\Lambda(r)$  is generated by  $\{y \in \Lambda^{\text{nd}}(0 : l_1, \dots, l_r) \mid l_1 + \dots + l_r = m\}$ . On the other hand,  $F_m\Lambda(r+1)/F_{m-1}\Lambda(r+1) = \bigoplus_{\substack{y \in \Lambda^{\text{nd}}(0 : l_1, \dots, l_r) \\ l_1 + \dots + l_r = m}} V_*^y$ , where  $V_*^y$  is generated by

$\{x \in \Lambda^{\text{nd}}(0 : l'_1, \dots, l'_r) \mid \underline{x \circ_{r+1} \zeta_{l'_{r+1}}} = y\}$ . (20) maps  $\mathbb{R}[y] \otimes \Gamma_*^r$  to  $V_*^y$ , and it is easy to see that this is a quasi-isomorphism for every  $y$ . Therefore, (20) is a quasi-isomorphism.  $\square$

By induction on  $r$ , we show that  $\Phi^{\mathcal{G}(r)} : \mathcal{G}(r) \rightarrow H_*(\Lambda(r))$  is an isomorphism. This is directly checked for  $r = 0, 1$ . Suppose that  $\Phi^{\mathcal{G}(r)}$  is an isomorphism for some  $r \geq 1$ . By Lemma 11.10,  $\Phi^{\mathcal{G}(r+1)}$  is surjective, and  $\dim H_*(\Lambda(r+1)) = (r+1)\dim H_*(\Lambda(r))$ . On the other hand,  $\dim \mathcal{G}(s) = \dim H_*(\mathcal{D}(s)) = s!$  for any  $s \geq 1$  (see [2] and [10]). Therefore  $\dim \mathcal{G}(r+1) = \dim H_*(\Lambda(r+1))$ . Thus,  $\Phi^{\mathcal{G}(r+1)}$  is an isomorphism. This completes the proof that  $\Phi^{\mathcal{G}}$  is an isomorphism.

Finally, we show that  $\Phi^{\mathcal{BV}}$  is an isomorphism. For every  $r \geq 1$ ,

$$(21) \quad \mathcal{G}(r) \otimes \mathcal{BV}(1)^{\otimes r} \rightarrow \mathcal{BV}(r); \quad x \otimes y_1 \otimes \dots \otimes y_r \mapsto (\dots((x \circ_1 y_1) \circ_2 y_2) \dots) \circ_r y_r$$

is an isomorphism, since a map between topological spaces

$$\mathcal{D}(r) \times f\mathcal{D}(1)^{\times r} \rightarrow f\mathcal{D}(r); \quad (x, y_1, \dots, y_r) \mapsto (\dots((x \circ_1 y_1) \circ_2 y_2) \dots) \circ_r y_r$$

is a homotopy equivalence. On the other hand, the next lemma holds.

**Lemma 11.11.** *For every  $r \geq 1$ , a chain map*

$$\Theta^r : \Lambda(r) \otimes f\Lambda(1)^{\otimes r} \rightarrow f\Lambda(r); \quad x \otimes y_1 \otimes \dots \otimes y_r \mapsto (\dots((x \circ_1 y_1) \circ_2 y_2) \dots) \circ_r y_r$$

*is a quasi-isomorphism.*

**Proof.** For any  $x \in f\Lambda^{\text{nd}}(0 : l_1, \dots, l_r)$ , let  $x'$  denote the decorated cactus obtained by forgetting the framing of  $x$ . For every integer  $m \geq 0$ , let  $F_m f\Lambda(r)$  denote the subspace of  $f\Lambda(r)$ , which is generated by  $\{x \in f\Lambda^{\text{nd}}(0 : l_1, \dots, l_r) \mid w(x') \leq r+m\}$ . This is a subcomplex of  $f\Lambda(r)$ . We set  $F_{-1}f\Lambda(r) := 0$ .

It is easy to see that, for every  $m \geq 0$  there holds  $\Theta^r(F_m\Lambda(r) \otimes f\Lambda(1)^{\otimes r}) \subset F_m f\Lambda(r)$ , and the map on quotients

$$(F_m\Lambda(r)/F_{m-1}\Lambda(r)) \otimes f\Lambda(1)^{\otimes r} \rightarrow F_m f\Lambda(r)/F_{m-1}f\Lambda(r)$$

is a quasi-isomorphism. Therefore,  $\Theta^r$  is a quasi-isomorphism.  $\square$

We need to show that  $\Phi^{\mathcal{BV}}(r) : \mathcal{BV}(r) \rightarrow H_*(f\Lambda(r))$  is an isomorphism. This is directly checked for  $r = 0, 1$ . Since  $\Phi^{\mathcal{G}}(r)$  is an isomorphism, the isomorphism (21) and Lemma 11.11 imply that  $\Phi^{\mathcal{BV}}(r)$  is an isomorphism for every  $r \geq 1$ .

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